

Dynamic Consistency and Regret*

Frank N. Caliendo[†]
Utah State University
frank.caliendo@usu.edu

T. Scott Findley
Utah State University
tscott.findley@usu.edu

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Abstract

Caplin and Leahy (2004) propose a major shift in how economists should think about welfare analysis. They demonstrate that the revealed preference welfare criterion is better suited to static analysis and is awkward in dynamic settings because it implies that policy makers would arbitrarily favor the most impatient decision rules from the Pareto set. To measure the economic significance of Caplin and Leahy's critique, we extend their conceptual framework to a quantitative setting of life-cycle consumption and saving. We find that using the revealed preference measure understates the socially optimal level of savings by a factor of 2 at our preferred parameterization. This is because the revealed preference criterion ignores the regret that rational individuals naturally feel about their past saving decisions. We relate our findings to the Fundamental Theorems of Welfare Economics, and we also argue that the standard rational framework is in fact consistent with phenomena (regret and undersaving) that are widely used as motivation to abandon it in favor of psychology-based theories like hyperbolic discounting.

Key words: Dynamic Consistency, Regret, Rational Choice, Retirement Saving.

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[†]Corresponding author.

1 Introduction

At least since Samuelson (1938, 1953) and Houthakker (1950), economists have relied on the principle of revealed preference as a guide in conducting normative economic analysis. That is, different public policy proposals should be evaluated through the lens of whatever utility function is used by individuals in their private decision making. For instance, if individuals discount the future at rate x in their private choices, then so should the government in its evaluation of public projects that involve intertemporal tradeoffs. Similarly, if individuals decide to save very little for retirement, then the government should assume this behavior is welfare maximizing and should not second guess the choice to live a relatively expensive lifestyle when young at the cost of an impoverished lifestyle when old.

Caplin and Leahy (2004) offer a provocative—but stunningly underappreciated—critique of revealed preference as the welfare criterion in dynamic settings. They argue that it “rests on faith, not logic” (p.1257), and they propose a major shift in how economists should think about welfare analysis by illustrating that the revealed preference welfare criterion is well-suited to static analysis but is awkward in dynamic settings. They make this point by considering the ideal consumption-saving allocations of a self of a given age, under the assumption that he has full control rights over past, present, and future resources. Each such solution represents a Pareto optimum, or an element from the Pareto set (in continuous time, a cross-section from the Pareto surface). They then show that revealed preference welfare analysis is based (arbitrarily) on the most impatient consumption-saving allocation from this Pareto set. The Pareto optima in the Pareto set are all identical only in the special, knife-edge case where the individual cares exponentially more about the past than the present. For all other cases in which past consumption is (weakly) discounted relative to present consumption—anywhere between no discounting to infinite discounting—later selves will wish that earlier selves had saved more even though actual, forward-looking decision making is dynamically consistent. Hence, regret over past decisions is a core feature of rational, dynamically-consistent decision making.

Essentially, Caplin and Leahy are careful to separate the concept of dynamically consistent *choice* from dynamically consistent *preferences*. While it is commonly understood that the standard rational model with exponential discounting involves consistent *choice* (Strotz (1956)), it has almost gone unnoticed (until Caplin and Leahy) that the standard model features inconsistent *preferences*, except in the special case where the individual cares exponentially more about the past than the present. Caplin and Leahy conclude that “revealed preference has been oversold” (p. 1267) and is possibly inappropriate altogether in a dynamic setting, because the choices that individuals make over time maximize welfare from

the perspective of time zero but not from any other vantage point, even though these choices are time consistent.

Because the Caplin-Leahy study is entirely theoretical (conceptual), the purpose of our paper is to provide *quantitative* confirmation that their ideas are economically significant. We consider a textbook problem featuring dynamically-consistent consumption-saving choices over the life cycle. We set our model in continuous time to maximize the frequency of choice, which allows us to highlight—in the most general way possible—internal disagreements concerning resource allocation among the many time-dated selves of a standard, rational individual. The individual in the model makes a plan and sticks with it, and yet we show that he deeply regrets his past saving decisions for any degree of discounting of past utility.

We find that using the revealed preference measure could understate the socially optimal level of savings by a factor of 2. This is because the revealed preference criterion ignores the regret that rational individuals naturally feel about past saving decisions. For instance, in our preferred parameterization which discounts past and future selves symmetrically, a 65 year old at the date of retirement will wish that he had saved about 3 times more than he actually does save. These discrepancies exist despite the fact that the individual's choices never deviate from his original consumption-saving plan.

The fact that dynamic consistency and regret coexist in the standard, rational model is not only theoretically interesting but also has first-order policy implications. Thinking of the individual as a collection of distinctly different time-dated selves, each with his own judgment about resource allocation over the life cycle, we can speak of the welfare theorems. The first theorem holds because the rational model leads to a consumption-saving allocation that is Pareto optimal. The second one definitely does not hold though, because lump-sum transfers of resources cannot support *any* target Pareto allocation: in the absence of any market frictions (like limitations on credit), the rational individual will not adjust his consumption in response to such transfers (as long as the net present value is zero) and therefore only the most impatient allocation from the Pareto set is attainable. The asymmetry in power among the selves, that arises from their temporal ordering, prevents the second welfare theorem from holding.

Of course, a more empirically relevant issue is whether lump-sum transfers are desirable when markets (like the credit market) are imperfect. An example would be a credit spread between the interest rates on borrowing and saving. In this case, transfers can successfully distort consumption and saving allocations in a way that could please later selves. This allows us to think about the mandatory saving role of social security without invoking the old paternalism arguments, or even the new paternalism arguments for that matter. This is

a fundamental shift away from the conventional wisdom that (i) a rational time-consistent individual (living in a dynamically efficient economy) will never benefit from mandatory saving, and that (ii) economic theory must move beyond rational choice in order to justify the popularity of mandatory saving programs.¹ Our analysis shows that neither is necessarily the case.

Finally, while our main analysis focuses on the consumption-saving decision, in a technical appendix we extend our model to include the decision of when to retire. We show that the presence of the retirement decision dramatically amplifies the significance of the Caplin-Leahy critique. Dynamic inconsistency among the time-dated, rational selves over their preferences for wealth accumulation is compounded because individuals wish that they had saved more to enjoy higher levels of future consumption *and* to finance an earlier retirement.

2 Four Optimization Problems

We purposefully focus on a simple, finite-horizon model with no uncertainty about income, returns, or longevity. In this setting, the existence of regret is not simply due to bad realizations from some stochastic process but is instead a core feature of rational, dynamically-consistent choice.

Concerning notation: $c(t)$ is consumption, $k(t)$ is savings, r is the interest rate on savings, $y(t)$ is disposable income, and \bar{T} is the lifespan. The forward-looking (prospective) discount function for a delay of length τ is $F(\tau)$ and the backward-looking (retrospective) discount function is $B(\tau)$. Note that we use the term “delay” to mean the absolute value of the length of time between the current moment and some other moment, whether that other moment is in the future or in the past. We set $F(\tau) = e^{-\rho_F \tau}$ and $B(\tau) = e^{-\rho_B \tau}$. Period utility is of the isoelastic variety, $u(c) = c^{1-\sigma}/(1-\sigma)$.

2.1 Problem 1: The Canonical Problem

Standing at time zero, the individual solves

$$\max : \int_0^{\bar{T}} e^{-\rho_F t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt, \quad (1)$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad (2)$$

¹Of course, social security taxation is not lump-sum, so the labor distortions imposed by such taxation would trade against any welfare gains that accrue to later selves.

$$k(0) = k(\bar{T}) = 0. \quad (3)$$

Using the Maximum Principle it is straightforward to show that the solution to this problem for $t \in [0, \bar{T}]$ is

$$c_0^*(t) = \frac{\int_0^{\bar{T}} y(t)e^{-rt} dt}{\int_0^{\bar{T}} e^{-\rho_F t/\sigma + (1-\sigma)rt/\sigma} dt} e^{(r-\rho_F)t/\sigma}, \quad k_0^*(t) = \int_0^t [y(s) - c_0^*(s)] e^{r(t-s)} ds. \quad (4)$$

2.2 Problem 2: Dynamic Consistency (Strotz (1956))

Suppose the individual has followed the initial plan from time zero up to some point v . If he were to reoptimize at vantage point v , he would behave according to

$$\max : \int_v^{\bar{T}} e^{-\rho_F(t-v)} \frac{c(t)^{1-\sigma}}{1-\sigma} dt, \quad (5)$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad (6)$$

$$k(v) = \int_0^v [y(t) - c_0^*(t)] e^{r(v-t)} dt, \quad k(\bar{T}) = 0. \quad (7)$$

Again, with the Maximum Principle and some algebra it is straightforward to show (as in Strotz (1956)) that for $t \in [v, \bar{T}]$

$$c_v^*(t) = c_0^*(t), \quad k_v^*(t) = k_0^*(t). \quad (8)$$

2.3 Problem 3: Full Control Rights (á la Caplin and Leahy (2004))

Suppose the individual is standing at age v , but unlike the previous problem he calculates the *entire* life-cycle consumption-saving program that he views as optimal. That is, at age v he ignores the reality of his current asset balance, inherited from himself through past saving decisions. Instead, he imagines what could have been: he computes the ideal plan for his past, present, and future as if he has full control rights over all lifetime resources. This program is the solution to a two-stage control problem

$$\max : \int_0^v e^{-\rho_B(v-t)} \frac{c(t)^{1-\sigma}}{1-\sigma} dt + \int_v^{\bar{T}} e^{-\rho_F(t-v)} \frac{c(t)^{1-\sigma}}{1-\sigma} dt, \quad (9)$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad (10)$$

$$k(0) = k(\bar{T}) = 0. \quad (11)$$

Following the Two-Stage Maximum Principle (Tomiya (1985)), we form a pair of Hamiltonians

$$\mathcal{H}_1 = e^{-\rho_B(v-t)} \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_1(t)[rk(t) + y(t) - c(t)], \text{ for } t \in [0, v], \quad (12)$$

$$\mathcal{H}_2 = e^{-\rho_F(t-v)} \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_2(t)[rk(t) + y(t) - c(t)], \text{ for } t \in [v, \bar{T}]. \quad (13)$$

The first-order conditions include

$$\frac{\partial \mathcal{H}_1}{\partial c(t)} = e^{-\rho_B(v-t)} c(t)^{-\sigma} - \lambda_1(t) = 0, \text{ for } t \in [0, v], \quad (14)$$

$$\frac{\partial \mathcal{H}_2}{\partial c(t)} = e^{-\rho_F(t-v)} c(t)^{-\sigma} - \lambda_2(t) = 0, \text{ for } t \in [v, \bar{T}], \quad (15)$$

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial \mathcal{H}_1}{\partial k(t)} = -r\lambda_1(t), \text{ for } t \in [0, v], \quad (16)$$

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial \mathcal{H}_2}{\partial k(t)} = -r\lambda_2(t), \text{ for } t \in [v, \bar{T}], \quad (17)$$

$$\lambda_1(v) = \lambda_2(v). \quad (18)$$

Solving the costate equations gives

$$\lambda_1(t) = a_1 e^{-rt}, \text{ for } t \in [0, v], \quad (19)$$

$$\lambda_2(t) = a_2 e^{-rt}, \text{ for } t \in [v, \bar{T}], \quad (20)$$

for constants a_1 and a_2 . The matching condition implies $a_1 = a_2$ and hence we can drop the subscripts

$$\lambda(t) = a e^{-rt}, \text{ for } t \in [0, \bar{T}]. \quad (21)$$

Rewrite the Maximum Conditions

$$c(t) = [a e^{\rho_B(v-t)-rt}]^{-1/\sigma}, \text{ for } t \in [0, v], \quad (22)$$

$$c(t) = [a e^{\rho_F(t-v)-rt}]^{-1/\sigma}, \text{ for } t \in [v, \bar{T}]. \quad (23)$$

From the state equation we have

$$k(t) = \begin{cases} \int_0^t \{y(s) - [ae^{\rho_B(v-s)-rs}]^{-1/\sigma}\} e^{r(t-s)} ds, & \text{for } t \in [0, v], \\ k(v)e^{r(t-v)} + \int_v^t \{y(s) - [ae^{\rho_F(s-v)-rs}]^{-1/\sigma}\} e^{r(t-s)} ds, & \text{for } t \in [v, \bar{T}]. \end{cases} \quad (24)$$

Evaluate $k(t)$ at $t = \bar{T}$ and use $k(\bar{T}) = 0$

$$\begin{aligned} 0 &= \int_0^v \{y(s) - [ae^{\rho_B(v-s)-rs}]^{-1/\sigma}\} e^{r(v-s)} ds \times e^{r(\bar{T}-v)} \\ &\quad + \int_v^{\bar{T}} \{y(s) - [ae^{\rho_F(s-v)-rs}]^{-1/\sigma}\} e^{r(\bar{T}-s)} ds \\ &= \int_0^v \{y(s) - [ae^{\rho_B(v-s)-rs}]^{-1/\sigma}\} e^{-rs} ds + \int_v^{\bar{T}} \{y(s) - [ae^{\rho_F(s-v)-rs}]^{-1/\sigma}\} e^{-rs} ds \end{aligned} \quad (25)$$

and then solve for a

$$a^{-1/\sigma} = \frac{\int_0^{\bar{T}} y(s)e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)-rs}]^{-1/\sigma} e^{-rs} ds + \int_v^{\bar{T}} [e^{\rho_F(s-v)-rs}]^{-1/\sigma} e^{-rs} ds}. \quad (26)$$

Hence, looking backward over $t \in [0, v]$ the optimal consumption path is

$$c_v^{**}(t) = \frac{\int_0^{\bar{T}} y(s)e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)-rs}]^{-1/\sigma} e^{-rs} ds + \int_v^{\bar{T}} [e^{\rho_F(s-v)-rs}]^{-1/\sigma} e^{-rs} ds} [e^{\rho_B(v-t)-rt}]^{-1/\sigma}, \quad (27)$$

and looking ahead over $t \in [v, \bar{T}]$ the optimal consumption path is

$$c_v^{**}(t) = \frac{\int_0^{\bar{T}} y(s)e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)-rs}]^{-1/\sigma} e^{-rs} ds + \int_v^{\bar{T}} [e^{\rho_F(s-v)-rs}]^{-1/\sigma} e^{-rs} ds} [e^{\rho_F(t-v)-rt}]^{-1/\sigma}. \quad (28)$$

Numerical examples illustrate the quantitative magnitude of the regret that the individual will experience from vantage point v . We set $\bar{T} = 55$ to reflect an economic lifespan from ages 25 to 80. Assuming retirement occurs after 40 years of work, we set $v = 40$ to capture the preferences of an individual at the date of retirement. We assume $y(t) = 1$ before 40 and $y(t) = 0.4$ afterwards to reflect social security benefits and other sources of income such as part-time work. We set $r = 1\%$ to align with typical risk-free returns on US treasuries and we set $\sigma = 1$. Finally, we assume a modest amount of forward discounting, $\rho_F = 2\%$.

All that remains is to select the backward discount rate ρ_B . We break into four numerical examples that vary in the value that we assign to ρ_B . In each example, the individual makes

a dynamically-consistent consumption-saving plan in the sense that he will always stick with his initial plan. At a knife-edge parameterization (Example 1 below), he will not regret his past choices. Yet, at all other parameterizations (e.g., Examples 2-4 below), he will regret his past choices and will wish that he had saved more over the interval $[0, v]$.

Example 1 ($\rho_B = -\rho_F$). For this knife-edge case, Problems 1 and 3 are mathematically equivalent and there is no regret about past choices. But notice that this requires a strong, counterfactual assumption that an individual cares exponentially *more* about the past than he cares about the present.

Example 2 ($\rho_B = \rho_F$). Here, the present is salient and the individual discounts at the same rate over all delays, whether forward or backward. The memory of a great vacation one year ago provides the same utility as the anticipation of a great vacation one year from now. This is our preferred parameterization.

Example 3 ($\rho_B = 0$). The individual cares just as much about each point in the past as he cares about the present.

Example 4 ($\rho_B = \infty$). The individual derives no utility whatsoever from past consumption and only cares about the present and future.

Figure 1 illustrates how a 65-year-old on the verge of retirement would value allocations over time. He discounts the future according to the forward discount function $e^{-\rho_F(t-v)}$, while he discounts past consumption according to the backward discount function $e^{-\rho_B(v-t)}$. We have plotted four different possibilities for the backward discount function (referred to as Examples 1-4 above).

In Figure 2 we plot the optimal consumption path $c_v^{**}(t)$ from the vantage point of retirement (i.e., v is set to model time 40, or age 65). This is the path that the individual would choose for his entire life cycle, if he had full control over his lifetime resources. Of course, he does not have full control; he is stuck with the endowment left by previous selves. Because he discounts the future exponentially, he will indeed stick with his initial consumption plan $c_0^*(t)$ no matter how he discounts the past. But notice how much differently he would like to have consumed if he could rewrite the past. Only in the knife-edge (unrealistic) case in which he cares exponentially *more* about the past than the present would he agree with the decisions of his past selves.

Figure 3 is similar to Figure 2, but now we plot optimal savings $k_v^{**}(t)$ from the vantage point of an individual at retirement. Again, only in the knife-edge case in which the individual cares exponentially more about the past than the present would he feel that he had saved

just the right amount for retirement. In this case, $k_v^{**}(t) = k_0^*(t)$. But for the other, more realistic examples in which the individual cares less about the past than the present ($\rho_B \geq 0$), he will experience significant regret. For instance, in our preferred parameterization where the individual discounts the past at the same rate that he discounts the future ($\rho_B = \rho_F$), he will wish that he had saved three times more than he actually does save. In fact, as long as $\rho_B > -\rho_F$, the individual will regret having saved “too little.” Even in the case where the individual doesn’t discount the past at all ($\rho_B = 0$), he will still wish that he had saved about twice as much as he actually does save. These findings cut sharply against claims that the rational, dynamically-consistent paradigm is incompatible with data on regret and undersaving.

2.4 Problem 4: Welfare (Caplin and Leahy (2004))

The solution consumption-saving allocation from the previous problem is optimal from the perspective of age v , but not from any other age. Hence, there is disagreement among the selves concerning how allocations should be valued. Each solution represents a Pareto optimum, or a cross-section of the Pareto surface.

Utility from the perspective of the individual standing at age v is

$$U(v) \equiv \int_0^v e^{-\rho_B(v-t)} u(c(t)) dt + \int_v^{\bar{T}} e^{-\rho_F(t-v)} u(c(t)) dt. \quad (29)$$

“Social welfare” embodies equal (utilitarian) treatment of the preferences of all the different selves,

$$\mathcal{SW} \equiv \int_0^{\bar{T}} U(v) dv. \quad (30)$$

Thus, the socially optimal consumption profile solves the following control problem

$$\max : \int_0^{\bar{T}} \left(\int_0^v e^{-\rho_B(v-t)} u(c(t)) dt + \int_v^{\bar{T}} e^{-\rho_F(t-v)} u(c(t)) dt \right) dv, \quad (31)$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad (32)$$

$$k(0) = k(\bar{T}) = 0. \quad (33)$$

This has the appearance of an intractable control problem, but we can make progress with some algebra.

Rewrite the objective functional

$$\begin{aligned}
& \int_0^{\bar{T}} \int_0^v e^{-\rho_B(v-t)} u(c(t)) dt dv + \int_0^{\bar{T}} \int_v^{\bar{T}} e^{-\rho_F(t-v)} u(c(t)) dt dv \\
&= \int_0^{\bar{T}} e^{-\rho_B v} \left(\int_0^v e^{\rho_B t} u(c(t)) dt \right) dv + \int_0^{\bar{T}} e^{\rho_F v} \left(\int_v^{\bar{T}} e^{-\rho_F t} u(c(t)) dt \right) dv. \quad (34)
\end{aligned}$$

Integrate by parts

$$\begin{aligned}
\int_0^{\bar{T}} e^{-\rho_B v} \left(\int_0^v e^{\rho_B t} u(c(t)) dt \right) dv &= \left[\frac{1}{-\rho_B} e^{-\rho_B v} \left(\int_0^v e^{\rho_B t} u(c(t)) dt \right) \right]_0^{\bar{T}} + \int_0^{\bar{T}} \frac{1}{\rho_B} u(c(v)) dv \\
&= \frac{1}{-\rho_B} e^{-\rho_B \bar{T}} \left(\int_0^{\bar{T}} e^{\rho_B t} u(c(t)) dt \right) + \int_0^{\bar{T}} \frac{1}{\rho_B} u(c(t)) dt \\
&= \int_0^{\bar{T}} \frac{1}{\rho_B} (1 - e^{-\rho_B(\bar{T}-t)}) u(c(t)) dt. \quad (35)
\end{aligned}$$

$$\begin{aligned}
\int_0^{\bar{T}} e^{\rho_F v} \left(\int_v^{\bar{T}} e^{-\rho_F t} u(c(t)) dt \right) dv &= \left[\frac{1}{\rho_F} e^{\rho_F v} \left(\int_v^{\bar{T}} e^{-\rho_F t} u(c(t)) dt \right) \right]_0^{\bar{T}} + \int_0^{\bar{T}} \frac{1}{\rho_F} u(c(v)) dv \\
&= \frac{-1}{\rho_F} \left(\int_0^{\bar{T}} e^{-\rho_F t} u(c(t)) dt \right) + \int_0^{\bar{T}} \frac{1}{\rho_F} u(c(t)) dt \\
&= \int_0^{\bar{T}} \frac{1}{\rho_F} (1 - e^{-\rho_F t}) u(c(t)) dt. \quad (36)
\end{aligned}$$

Thus, the social optimization problem can be re-stated compactly

$$\max : \int_0^{\bar{T}} D(t) u(c(t)) dt, \quad (37)$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad (38)$$

$$k(0) = k(\bar{T}) = 0, \quad (39)$$

where $D(t)$ is the social discount function that assigns weights to the different selves

$$D(t) \equiv \frac{1}{\rho_B} (1 - e^{-\rho_B(\bar{T}-t)}) + \frac{1}{\rho_F} (1 - e^{-\rho_F t}). \quad (40)$$

The solution to this optimal control problem for isoelastic utility is

$$c_{SW}^*(t) = \frac{\int_0^{\bar{T}} y(t) e^{-rt} dt}{\int_0^{\bar{T}} e^{rt/\sigma} D(t)^{1/\sigma} e^{-rt} dt} e^{rt/\sigma} D(t)^{1/\sigma}, \quad (41)$$

$$k_{SW}^*(t) = \int_0^t [y(s) - c_{SW}^*(s)] e^{r(t-s)} ds. \quad (42)$$

As in the previous subsection, we continue to use the same parameterizations of the backward discount rate ρ_B in numerical examples.

Example 1 ($\rho_B = -\rho_F$). The social discount function is the same as the private, forward discount function:

$$\begin{aligned} D(t) &= \frac{1}{-\rho_F} (1 - e^{\rho_F(\bar{T}-t)}) + \frac{1}{\rho_F} (1 - e^{-\rho_F t}) \\ &= \frac{1}{\rho_F} [e^{\rho_F \bar{T}} - 1] e^{-\rho_F t} \\ &\propto e^{-\rho_F t}. \end{aligned} \quad (43)$$

At this knife-edge parameterization of the backward discount rate, the preferences of all the different selves are in perfect agreement concerning the valuation of allocations over time, and so the social discount function reflects this harmony.

Example 2 ($\rho_B = \rho_F$). In this case $D(t)$ is strictly concave (quadratic) with a peak at $t = \bar{T}/2$:

$$D(t) = \frac{1}{\rho} [2 - e^{-\rho(\bar{T}-t)} - e^{-\rho t}]. \quad (44)$$

$$D'(t) = \frac{1}{\rho} [-\rho e^{-\rho(\bar{T}-t)} + \rho e^{-\rho t}]. \quad (45)$$

$$D'(t) = 0 \iff t = \frac{\bar{T}}{2}. \quad (46)$$

$$D''(t) = \frac{1}{\rho} [-\rho^2 e^{-\rho(\bar{T}-t)} - \rho^2 e^{-\rho t}] < 0. \quad (47)$$

This function has a quadratic shape because the midpoint is, on average, the closest to all the various vantage points and hence it is discounted the least in an aggregate sense. On the other hand, the boundaries of the life cycle are, on average, the furthest from all the vantage points and so they get the least weight in the social optimization problem.

Example 3 ($\rho_B = 0$). Here $D(t)$ is strictly concave and strictly decreasing with a peak at

$t = 0$:

$$\begin{aligned}
\lim_{\rho_B \rightarrow 0} D(t) &= \lim_{\rho_B \rightarrow 0} \frac{1}{\rho_B} (1 - e^{-\rho_B(\bar{T}-t)}) + \lim_{\rho_B \rightarrow 0} \frac{1}{\rho_F} (1 - e^{-\rho_F t}) \\
&= \lim_{\rho_B \rightarrow 0} \frac{\frac{d}{d\rho_B} (1 - e^{-\rho_B(\bar{T}-t)})}{\frac{d}{d\rho_B} \rho_B} + \frac{1}{\rho_F} (1 - e^{-\rho_F t}) \\
&= \lim_{\rho_B \rightarrow 0} \frac{(\bar{T} - t)e^{-\rho_B(\bar{T}-t)}}{1} + \frac{1}{\rho_F} (1 - e^{-\rho_F t}) \\
&= \bar{T} - t + \frac{1}{\rho_F} (1 - e^{-\rho_F t}). \tag{48}
\end{aligned}$$

$$\frac{d}{dt} \left(\lim_{\rho_B \rightarrow 0} D(t) \right) = -1 + e^{-\rho_F t} < 0. \tag{49}$$

$$\frac{d^2}{dt^2} \left(\lim_{\rho_B \rightarrow 0} D(t) \right) = -\rho_F e^{-\rho_F t} < 0. \tag{50}$$

Individuals have perfect recall in the sense that a fun vacation at any point in the past is just as valuable today as a fun vacation today.

Example 4 ($\rho_B = \infty$). Here $D(t)$ is strictly concave and strictly increasing, with a peak at $t = \bar{T}$:

$$\begin{aligned}
D(t) &= \frac{1}{\infty} (1 - e^{-\infty(\bar{T}-t)}) + \frac{1}{\rho_F} (1 - e^{-\rho_F t}) \\
&= \frac{1}{\rho_F} (1 - e^{-\rho_F t}). \tag{51}
\end{aligned}$$

$$D'(t) = e^{-\rho_F t} > 0. \tag{52}$$

$$D''(t) = -\rho_F e^{-\rho_F t} < 0. \tag{53}$$

In this case individuals do not care at all about the past, but all of the selves care about utility when old (to varying degrees). The social discount function respects this “consensus” opinion about the importance of old-age consumption and places the most weight on utility when old.

Figure 4 plots the social discount function $D(t)$ that corresponds to each of the alternative examples of the backward discount function. Each social discount function has been normalized to ensure that it peaks at unity. Figure 5 plots socially optimal consumption $c_{\mathcal{S}\mathcal{W}}^*(t)$ for each of these cases, and Figure 6 reports the socially optimal savings profiles that result $k_{\mathcal{S}\mathcal{W}}^*(t)$. In our preferred parameterization ($\rho_B = \rho_F$), the social optimum requires twice as much savings by the date of retirement compared to what the rational individual

accumulates on his own. Even if the individual cares just as much about the past as he cares about the present ($\rho_B = 0$) and therefore prefers relatively high levels of consumption when young, it is still the case that the individual seriously undersaves for retirement. Finally, if the individual doesn't care at all about the past ($\rho_B = \infty$) then undersaving is the most pronounced.

3 Why Economists Should Care

We discuss two major issues that underscore the importance of Caplin and Leahy's critique of revealed preference in dynamic settings. The first reason is methodological: some empirical findings that have been used to motivate the need to leave rationality in favor of psychology-based theories are in fact consistent with the standard model of rationality once Caplin-Leahy's insights are understood. The second reason is practical: policy conclusions in dynamic settings will depend crucially on whether welfare analysis follows the revealed preference tradition or adopts the Caplin-Leahy approach.

3.1 Psychology-Based Theories

The practice of utilizing findings from the field of psychology to provide new insight into many important economic questions has become quite commonplace in economics research. In fact, much of the justification (in the field of Behavioral Economics) centers around the conjecture that the standard model of rationality does not embody the possibility of preference reversals. For example, Rabin (1998, p.38) states:

An important qualitative feature of exponential discounting is that it implies that a person's intertemporal preferences are time-consistent: A person feels the same about a given intertemporal tradeoff no matter when she is asked.

In his acceptance prize for winning the Nobel Prize in Economics, Akerlof (2002, p.422) similarly asserts:

For New Classical economics, saving too little or too much...is an impossibility, a straightforward contradiction of the assumptions of the model. Since saving is the result of individual utility maximization, it must, absent externalities, be just right...The hyperbolic discount function, which has been used to study intertemporal savings choices, can be used to formalize the distinction between the utility function that describes actual saving behavior and the utility function that measures the welfare resulting from that behavior.

Also, Camerer (1999, p.10577) states:

Behavioral economics can also provide a more realistic and thoughtful basis for making economic policy...For example, if people weight the future hyperbolically rather than exponentially, they will impulsively buy goods they will soon regret having bought.

For these and for other reasons, behavioral economists have moved beyond the standard model and looked to alternative theories of intertemporal choice, such as hyperbolic discounting.² Although we personally believe that making use of the hyperbolic discounting framework has led to advancements in how economists view the world, here we would like to point out that the standard model with exponential discounting has a lot more to offer than previously supposed. Indeed, it is our contention here that the Caplin-Leahy critique of revealed preference analogously implies that the standard framework of rationality used by economists is one in which preferences are in effect dynamically *inconsistent*, albeit choice itself is dynamically *consistent*. That is, the standard model of rationality is in fact compatible with evidence from psychology regarding non-stationary preferences, which is often manifest by the fact that people frequently experience the phenomenon of regret. To place this assertion in context, a model with hyperbolic discounting exhibits non-stationary preferences, although the question as to whether or not choice itself is dynamically consistent follows from what assumption is made by the researcher regarding the degree of awareness or cognition that the modeled individual possesses (i.e., naiveté versus sophistication).³ In sum, if choice data reflect the idea that an individual's actual choices deviate from his or her earlier plans, then a hyperbolic discounting model with naivete is the appropriate construct to employ. But, if a researcher wants to use a model that is compatible with the fact that a person simply regrets his or her past choices, then the standard model of rationality easily fits the bill.

In our analysis above, we have provided a quantitative confirmation to this idea within the realm of consumption and saving over the life cycle. The rational individual, whose choices are dynamically consistent, does indeed regret the level of savings that he inherits from himself as a result of his past saving decisions. He most certainly wishes that he had saved more in the past than what he actually saved. In the Technical Appendix, we have extended

²For similar statements that economists necessarily need to look beyond the standard model when examining non-stationary preferences, see Laibson (1997), Rabin (1998), O'Donoghue and Rabin (1999a), and Frederick, Loewenstein, and O'Donoghue (2002), among many others.

³Modeling a divergence between intentions and actual behavior requires that individuals naively fail to account for their own time inconsistency (O'Donoghue and Rabin 1999b, 1999c, 2000, 2003; Caillaud and Jullien 2000; Prelec 2004; Beshears et al. 2008; D'Orlando and Sanfilippo 2010; Herweg and Müller 2011).

this confirmation to the idea that preferences are dynamically inconsistent regarding the retirement decision (extensive labor supply margin). This compounds the magnitude of the dynamic inconsistency in preferences, since there is a second channel through which regret can affect the individual: not only does the individual wish that he had inherited more savings for a given date of retirement, but he also wishes that he had inherited more savings so that he could finance an earlier date of retirement. As demonstrated conceptually by Caplin and Leahy (2004) and as confirmed quantitatively in this paper for the stylized case of life-cycle consumption and saving, the fact that the standard model of dynamically-consistent choice is a model of dynamically-inconsistent preferences can breathe some new life into the standard model of rationality: it is much suited to study the concepts of regret and undersaving than previously supposed.

3.2 Social Security

In this section we illustrate that using the revealed preference welfare criterion can lead to a very different set of conclusions than when using the social welfare criterion of Caplin and Leahy. The case of Social Security is an excellent example of this. Consider a fully funded Social Security program to focus on the mandatory savings aspect of the program while abstracting from inefficiencies in financing the system. Also suppose labor is supplied inelastically to focus on the consumption-saving dimension of the life cycle.

For the theoretical case in which capital markets are complete and individuals can borrow and saving freely at the same interest rate, the fully funded Social Security will not affect the individual's consumption over the life cycle. The individual reduces saving one-for-one for every dollar paid in taxes, leaving consumption unchanged. In this case, it wouldn't matter how welfare is evaluated—whether it is according to the revealed preference criterion or the Caplin-Leahy social welfare criterion—the conclusion that Social Security is irrelevant. However, perfect capital markets is not an empirically reasonable assumption (Davis, Kubler, and Willen (2006)).

Suppose the interest rate on borrowing exceeds the interest rate on saving, and suppose the internal rate of return on Social Security matches the interest rate on private saving. For any point in the life cycle when the individual is a saver, an extra dollar in Social Security taxation will be offset one-for-one by a reduction in private saving. For any point when the individual is a borrower, an extra dollar in Social Security taxation would cause the individual to borrow less than one dollar in response. The Social Security tax in this case is financed by a combination of reduced consumption and increased debt. Hence, for the typical case in which the individual borrows during the early years of the life cycle when wages are

low and then begin saving for retirement during the middle age years, Social Security is bound to reduce consumption when young and increase consumption during the retirement years (Findley and Caliendo (2016)). From Figure 5, it is plain to see that the socially optimal consumption path requires just such a transfer, relative to the consumption that is actually followed. In other words, fully funded Social Security can potentially improve the welfare of a fully rational, time-consistent individual, when welfare is measured through the lens of Caplin-Leahy social welfare.

This important conclusion is orthogonal to what is prescribed by the revealed preference welfare criterion. In that case, the individual is already behaving perfectly optimally in the absence of any Social Security program. Forcing the individual to save at the market rate—when he is already borrowing at a high rate while young—is not useful at all. He already had the ability to do that if he wanted to, and forcing him to do it is strictly welfare reducing because the only way to unwind forced saving is by taking on high cost debt. Hence, fully funded Social Security is bad idea for rational consumers when revealed preference is the welfare standard.

All of the intuition described above for the case of a credit spread would carry over to the more extreme case of a hard borrowing constraint. If the individual is borrowing constrained during the early years when wages are low and consumption is equal to wages, Social Security taxation cannot be unwound and consumption must be reduced one-for-one with taxation in return for higher consumption later when benefits are collected. This transfer of resources can improve welfare when viewed through the lens of Caplin-Leahy social welfare, but can never improve welfare under the revealed preference criterion.

In sum, it is widely accepted that Social Security's mandatory saving role (as opposed to its risk sharing and redistributive roles) is incompatible with the rational, time-consistent model of life-cycle consumption and saving. The common view is that fully funded Social Security would either have no effect or a negative effect on individual welfare. As a result, the profession has turned its attention to psychology-based theories to rationalize mandatory saving through Social Security. Examples include hyperbolic discounting (Akerlof (1998), Laibson (1998), İmrohoroğlu, İmrohoroğlu, and Joines (2003), Gul and Pesendorfer (2004), DellaVigna (2009), Caliendo (2011), Cremer and Pestieau (2011), Guo and Caliendo (2014), Findley and Caliendo (2016)), impulsivity (Findley (2016)), short planning horizons (Findley and Caliendo (2009)), limited computational ability (Caliendo and Findley (2013)), and hand-to-mouth behavior (Feldstein (1985), Docquier (2002), Cremer et al. (2008), Caliendo and Gahramanov (2009)). And while some of these alternative models can indeed justify fully funded Social Security, to some extent they have all been motivated under the false premise that the rational, time-consistent model fails to justify Social Security. In reality,

it is not the model that has failed to justify Social Security, it is the revealed preference criterion that has failed to justify Social Security. Andersen and Bhattacharya (2011) make this point in a parallel fashion, by assuming that rational, borrowing constrained individuals benefit from mandatory saving when the social discount rate is less than the private discount rate.

A final example is Social Security reform. Here again, the revealed preference welfare criterion leads to a very different set of conclusions than the social welfare criterion of Caplin and Leahy. Consider the reform proposal of Hurst and Willen (2007), who proposed that young individual be exempt from Social Security taxation when young. Evaluated through the lens of the revealed preference welfare criterion, this reform could be welfare improving because it would help to relax the need to borrow at high interest rates when young, when wage income is quite low. The reform would provide consumption-smoothing gains by helping young individuals consume more. However, when viewed through the lens of Caplin and Leahy's social welfare metric, this reform works in exactly the wrong direction because the social welfare increases when the individual saves more than he naturally would on his own.

We conclude that the question of which welfare metric should be used to evaluate individual choice is much more than just an academic debate. It has first-order effects on basic policy questions such as whether Social Security is a good idea and how the government should reform Social Security. In our view, the time consistency of rational decision rules—in the sense that individuals stick to their previous plans—obscures the subtle but crucial point that rational individuals with full control rights over all lifetime resources (past, present, and future) have time-inconsistent preferences over consumption and saving allocations except in the knife edge case in which the individual cares exponential more about the past than the present. The time inconsistency of the full-control rights allocation problem is the source of the divergence in policy recommendations between revealed preference welfare and Caplin-Leahy welfare. And simply ignoring the Caplin-Leahy result in favor of revealed preference carries the baggage that policies will be designed arbitrarily around the desires of the time-zero self of an individual.

4 Concluding Discussion

Caplin and Leahy's paper forces economists to take a stand on how we treat the past in the calculation of welfare. While conventional welfare analysis seems to be silent in this issue, in fact it is not. The revealed preference welfare criterion tacitly assumes that all time-dated selves of a single individual agree on the ideal allocation. However, such harmony of views is achieved only if individual care exponentially more about the past than the present. The

revealed preference welfare criterion does not allow the research to side step the seemingly messy issue of how to weight past experience; instead, it forces the researcher to make the unnatural (and, we suspect, empirically empty) assumption that the distance past matters more than anything else.

The revealed preference welfare criterion is nested by Caplin and Leahy's more general method that allows the researcher to assign weights to the preferences of the different time-dated selves of an individual. Revealed preference puts all of the weight on the time-zero self. While there is nothing wrong with that assumption, philosophically it is no more compelling than any other weighting scheme.

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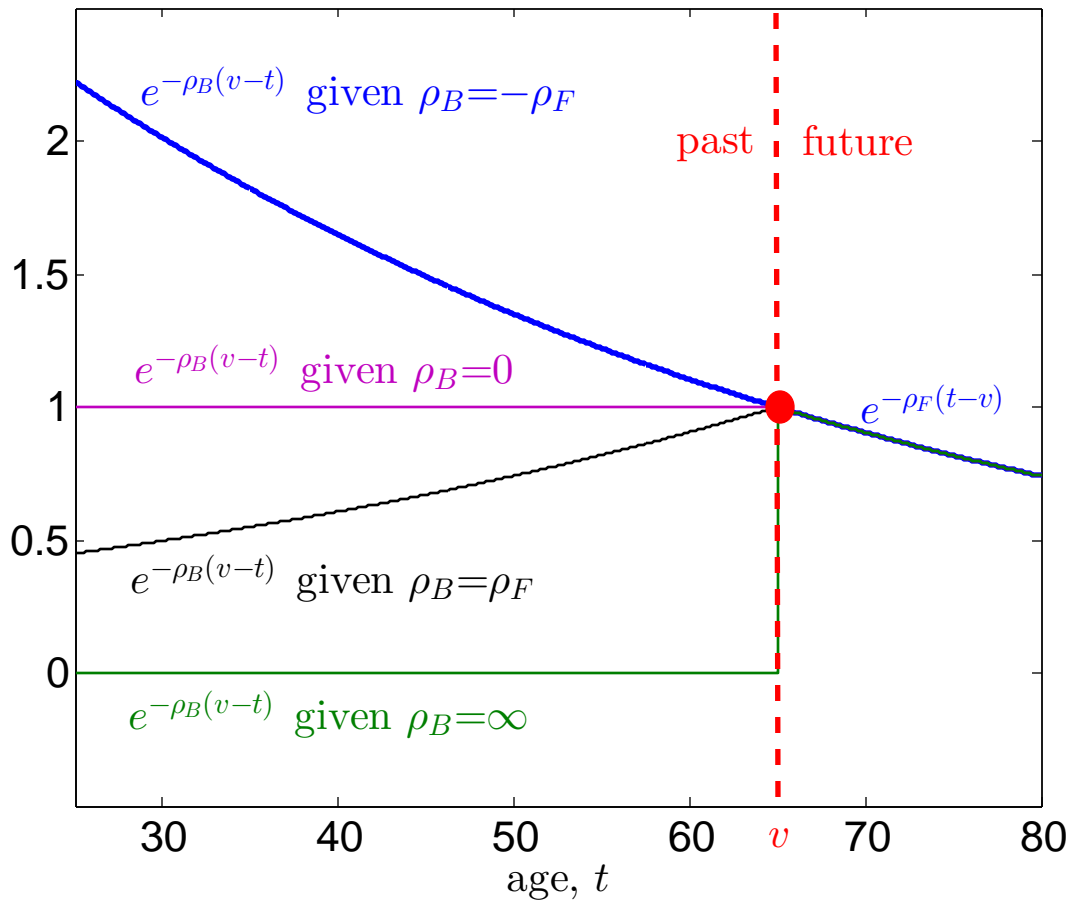
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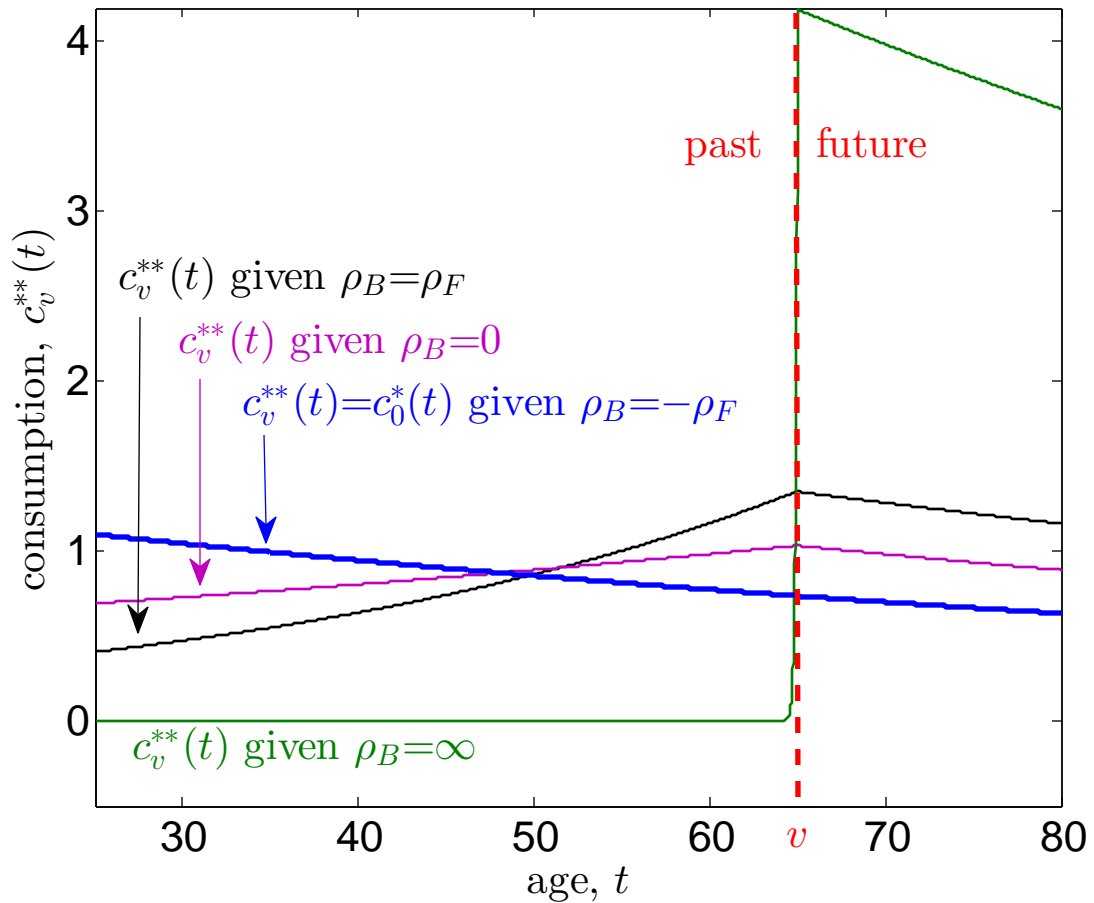
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Figure 1. Discount Functions from the Vantage Point of Retirement



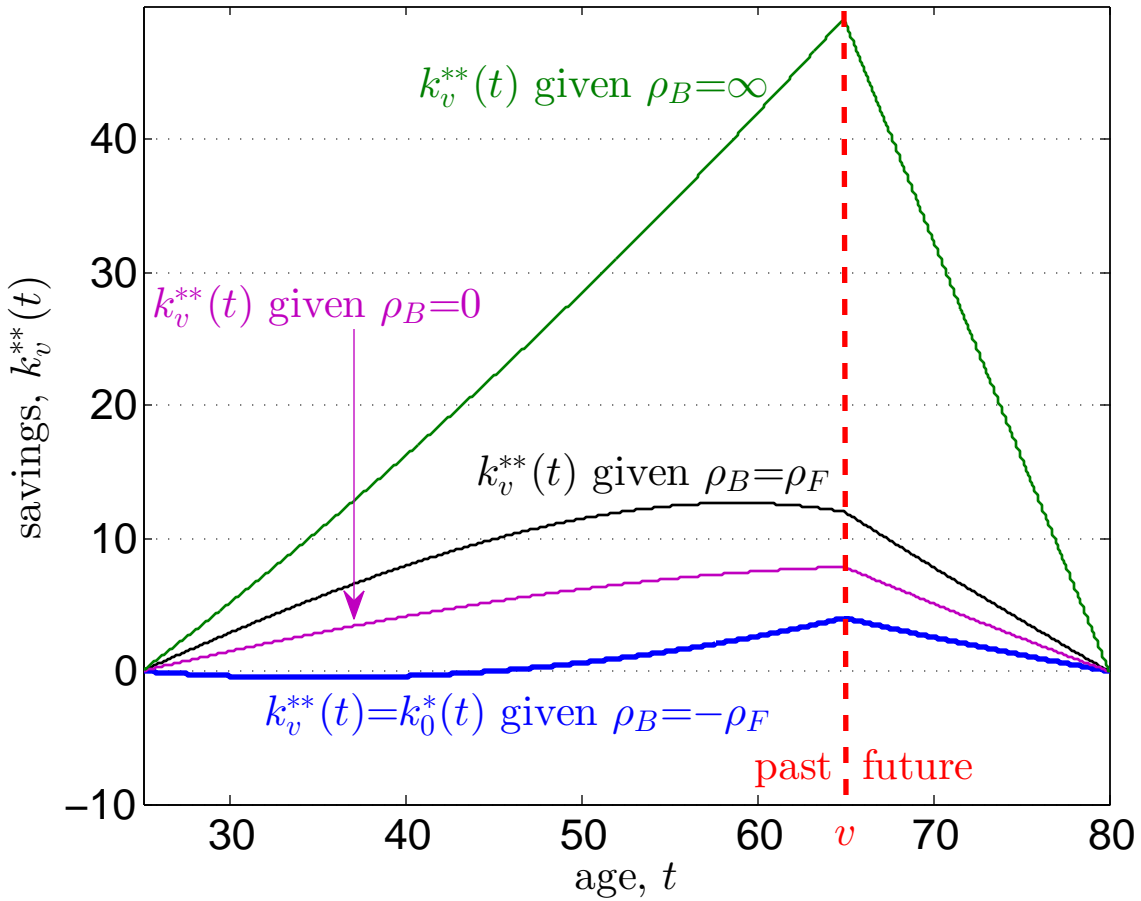
The forward discount rate, ρ_F , is set to 0.02.

Figure 2. Optimal Consumption from the Vantage Point of Retirement



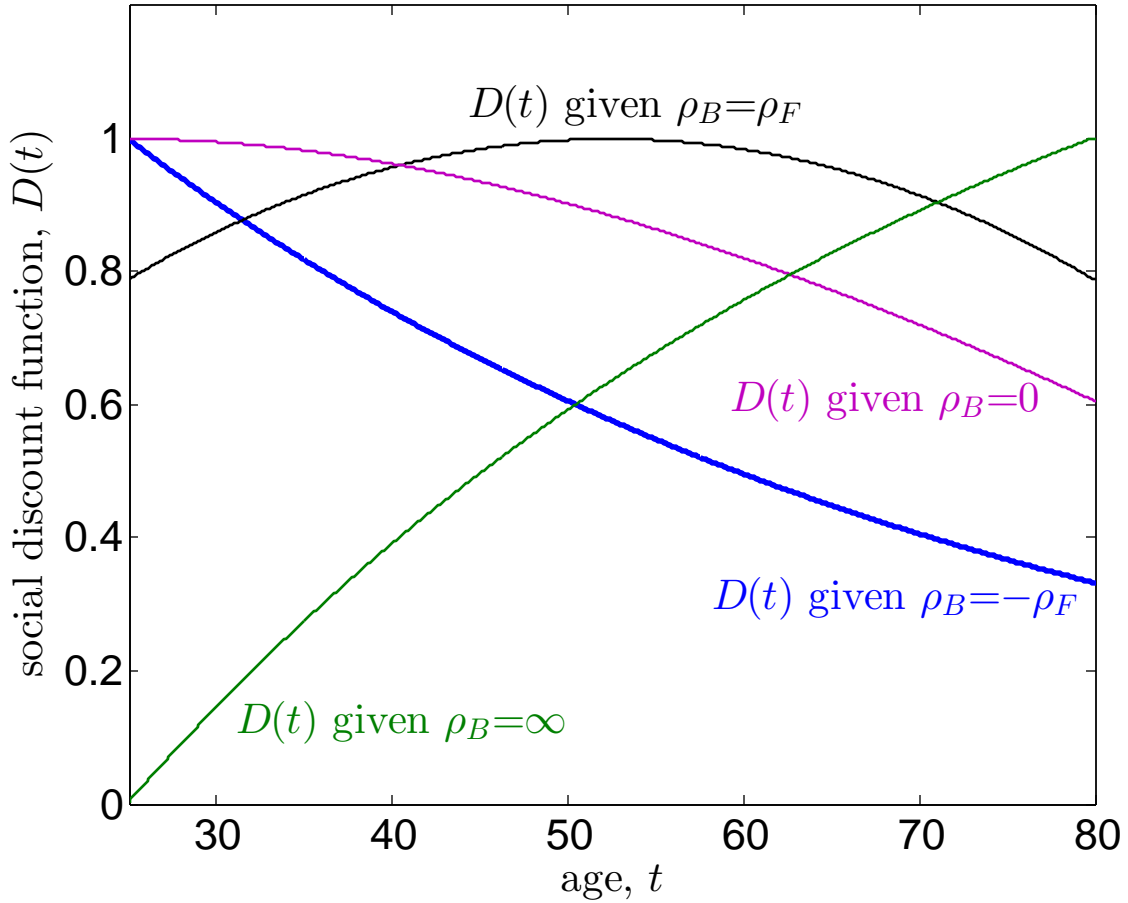
The forward discount rate, ρ_F , is set to 0.02.

Figure 3. Optimal Savings from the Vantage Point of Retirement



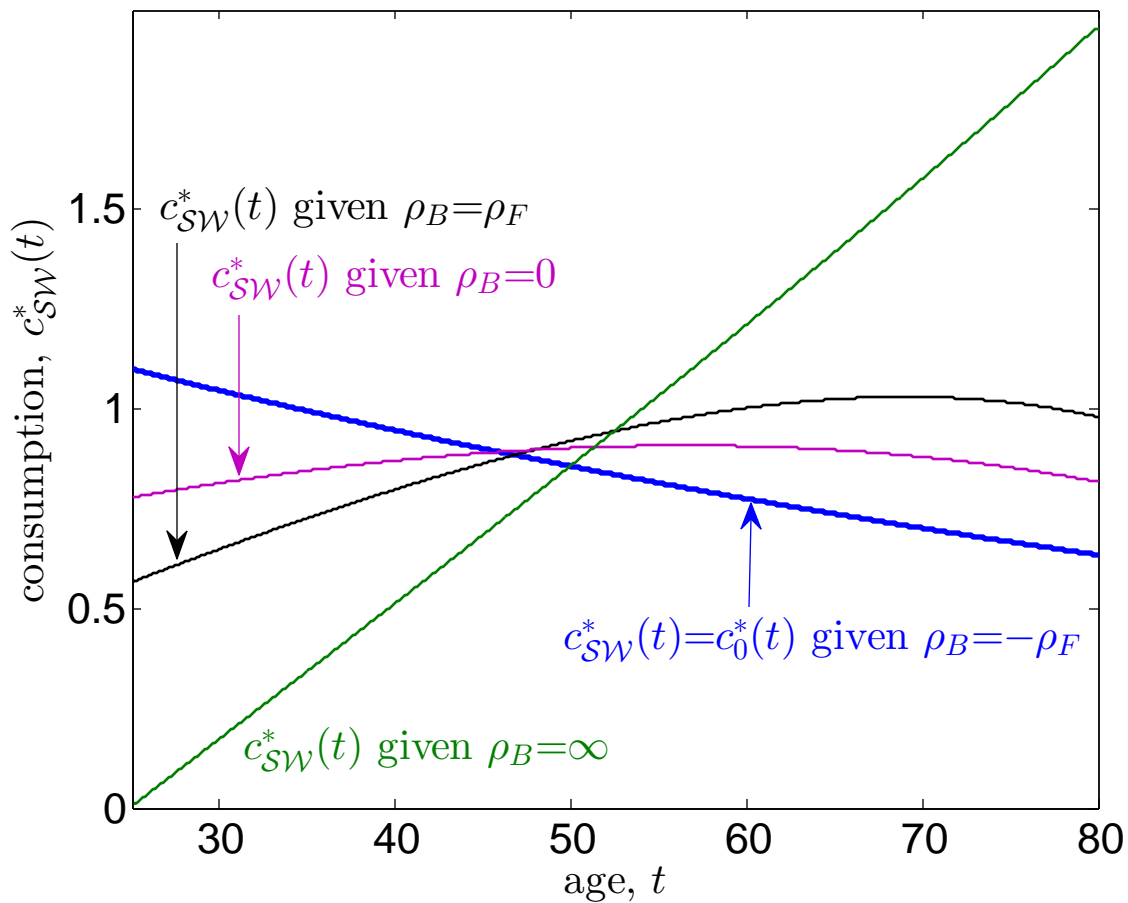
The forward discount rate, ρ_F , is set to 0.02.

Figure 4. Social Discount Functions



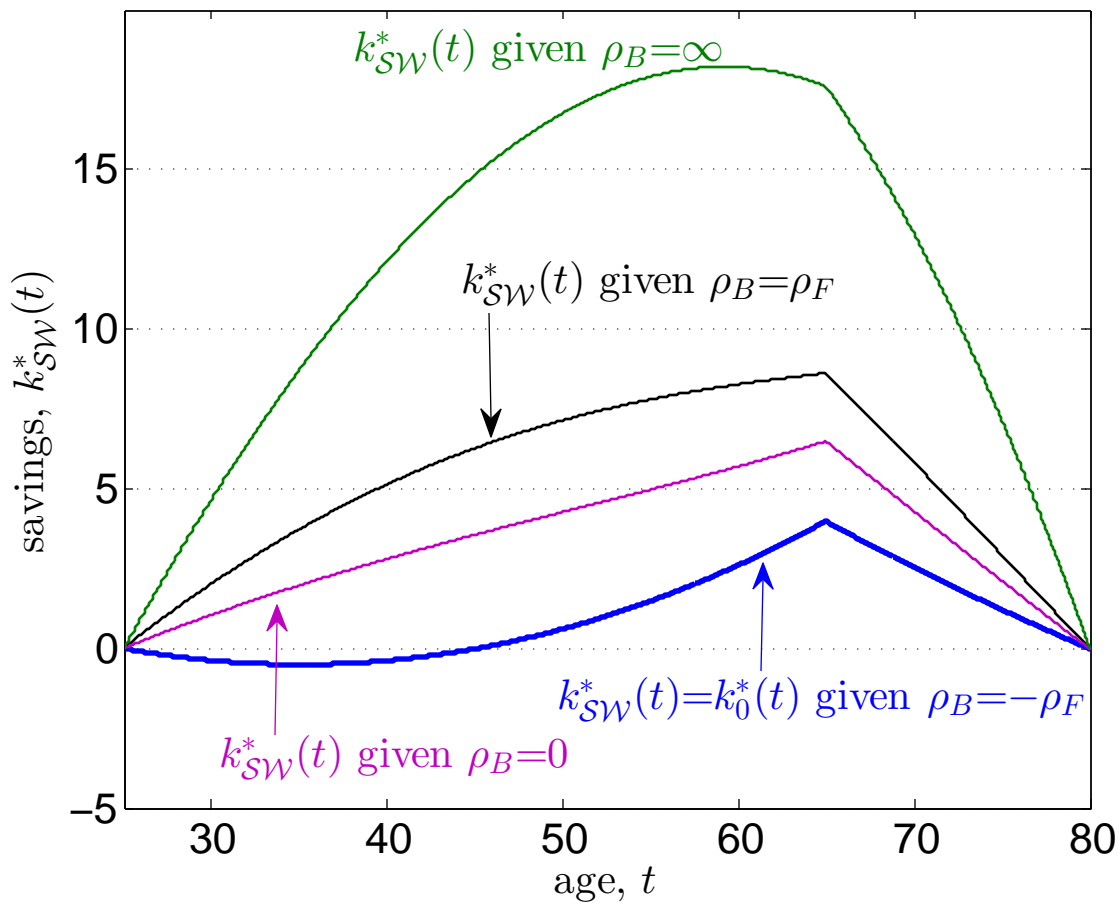
The forward discount rate, ρ_F , is set to 0.02. Each function is normalized to ensure a peak at unity.

Figure 5. Socially Optimal (Utilitarian) Consumption



The forward discount rate, ρ_F , is set to 0.02.

Figure 6. Socially Optimal (Utilitarian) Savings



The forward discount rate, ρ_F , is set to 0.02.

Not for Publication Technical Appendix to: Dynamic Consistency and Regret

This appendix includes:

1. Brief introductory remarks to highlight that: extending our quantitative model to include the decision of when to retire dramatically amplifies the economic significance of Caplin and Leahy's (2004) critique of revealed preference.
2. A model where individuals can choose between no work and full time work, with a fixed cost of work that delivers endogenous retirement.
3. Numerical examples to support our conjecture.

1 Brief Summary of Technical Appendix

In this appendix we extend the arguments in Caplin and Leahy (2004) to a life-cycle setting that includes the decision of when to retire. We show that the retirement margin has a first-order impact on the degree of regret that rational individuals feel about their past saving behavior. In other words, relying on the revealed preference argument to conclude that individuals save just the right amount for retirement is especially problematic when retirement is a choice in the model.

Dynamic inconsistency about one's preference for wealth accumulation is compounded in our model because individuals wish that they had saved more for two reasons: first, so that current and future selves can enjoy a higher level of consumption (just as in Caplin and Leahy), and second, to finance an earlier retirement. Quantitatively, this second channel leads to far more regret about undersaving than does the first channel in isolation.

To illustrate these points, we calibrate a textbook consumption-saving model so that the individual retires at age 66. This is the optimal retirement age from the perspective of his initial age (age 25), and it is the age at which he ultimately retires because he is a rational, time-consistent decision maker. However, as he gets older, he will begin to wish that he had saved more when younger and he would like to retire earlier than planned.

For example, when he hits the age of 40, he will be almost done repaying his debts and ready to begin saving for retirement. Yet he will wish that he had never borrowed anything at all, and instead he will wish that he was much further toward reaching his retirement savings goal. If the individual discounts past utility at the same rate that he discounts future utility, then he will wish that he had saved about 1/3 of his past income to allow him to retire early at age 54. If he doesn't discount the past at all, then he will wish that he had saved about 1/5 of his past income to allow him to retire early at age 58. And if he doesn't care at all about the past, then he will wish that he had saved every penny of his past income to allow him to retire immediately at age 40.

Importantly, when we decompose the disagreement about retirement savings among the time-dated selves into the part that would occur if retirement is fixed and the overall effect when retirement is endogenous, we find that the second effect is much larger than the first. In other words, the dynamic inconsistency in the individual's taste for wealth accumulation is significantly amplified by the presence of the retirement margin.

2 Model

2.1 Notation

We purposefully focus on a simple, finite-horizon setting with no uncertainty about income, returns, or longevity. In this setting, regret is not simply due to bad realizations from some stochastic process but is instead a core feature of the preference structure of the individual.

Concerning notation: t is time, $c(t)$ is consumption, $k(t)$ is the zero-interest savings account, $y(t)$ is disposable income, T is retirement, and \bar{T} is the lifespan. Before retirement the individual works full time and earns y_1 and after retirement he works either part time or not at all and earns $y_2 \in [0, y_1)$. Retirement is an irreversible decision. Period utility when working full time is $u(c) = c^{1-\sigma}/(1-\sigma) - \psi$ and period utility when retired is $u(c) = c^{1-\sigma}/(1-\sigma)$.

The forward looking discount function for a delay of length τ is $F(\tau)$ and the backward looking discount function is $B(\tau)$. Note that we use the term “delay” to mean the absolute value of the length of time between the current moment and some other moment, whether that other moment is in the future or in the past. The usual properties hold: $F(0) = B(0) = 1$, $F(\tau) > 0$, $B(\tau) > 0$, $F'(\tau) < 0$, $B'(\tau) < 0$.

2.2 Optimization with Full Control Rights (Caplin and Leahy (2004))

Suppose the individual is standing at vantage point $v \in [0, \bar{T}]$, but unlike a standard problem in which he takes existing assets at age v as given, he now imagines the entire life-cycle consumption-saving program that he views as optimal. That is, he ignores the reality of how much (or little) he has already accumulated in his asset account by age v and instead imagines what might have been: he imagines the ideal plan for his past, present, and future consumption and savings. This program is the solution to a multi-stage optimal control problem:

$$\max_{\{c(t), T\}} : \left[\begin{array}{l} \left(\int_0^v B(v-t) \frac{c(t)^{1-\sigma}}{1-\sigma} dt + \int_v^{\bar{T}} F(t-v) \frac{c(t)^{1-\sigma}}{1-\sigma} dt \right) \\ - \left(\int_0^{\min\{v, T\}} B(v-t) \psi dt + \int_v^{\max\{v, T\}} F(t-v) \psi dt \right) \end{array} \right],$$

subject to

$$\begin{aligned} \frac{dk(t)}{dt} &= y(t) - c(t), \\ y(t) &= \begin{cases} y_1, & \text{for } t \in [0, T], \\ y_2, & \text{for } t \in [T, \bar{T}], \end{cases} \\ k(0) &= k(\bar{T}) = 0. \end{aligned}$$

Notice that the upper limits on the third and fourth integrals in the objective functional must be flexible enough to take into account that a given vantage point v could lie before or after a given retirement age T .

We break this problem into an inner problem and an outer problem. The inner problem solves for the optimal consumption and savings for a fixed retirement date. The outer problem solves for the optimal retirement date.

2.2.1 The Inner Problem

In the inner problem, we can ignore the third and fourth integrals in the objective functional, so that the inner problem is a standard two-stage control problem. Form a pair of Hamiltonians

$$\begin{aligned} \mathcal{H}_1 &= B(v-t) \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_1(t)[y(t) - c(t)], \text{ for } t \in [0, v], \\ \mathcal{H}_2 &= F(t-v) \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_2(t)[y(t) - c(t)], \text{ for } t \in [v, \bar{T}]. \end{aligned}$$

The first-order conditions include

$$\frac{\partial \mathcal{H}_1}{\partial c(t)} = B(v-t)c(t)^{-\sigma} - \lambda_1(t) = 0, \text{ for } t \in [0, v],$$

$$\frac{\partial \mathcal{H}_2}{\partial c(t)} = F(t-v)c(t)^{-\sigma} - \lambda_2(t) = 0, \text{ for } t \in [v, \bar{T}],$$

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial \mathcal{H}_1}{\partial k(t)} = 0, \text{ for } t \in [0, v],$$

$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial \mathcal{H}_2}{\partial k(t)} = 0, \text{ for } t \in [v, \bar{T}],$$

$$\lambda_1(v) = \lambda_2(v).$$

Solving the costate equations together with the matching condition gives

$$\lambda_1(t) = \lambda_2(t) = \lambda, \text{ for } t \in [0, \bar{T}].$$

Rewrite the Maximum Conditions

$$c(t) = \lambda^{-1/\sigma} B(v-t)^{1/\sigma}, \text{ for } t \in [0, v],$$

$$c(t) = \lambda^{-1/\sigma} F(t-v)^{1/\sigma}, \text{ for } t \in [v, \bar{T}].$$

From the state equation we have

$$k(t) = \begin{cases} \int_0^t \left\{ y(s) - \lambda^{-1/\sigma} B(v-s)^{1/\sigma} \right\} ds, & \text{for } t \in [0, v], \\ k(v) + \int_v^t \left\{ y(s) - \lambda^{-1/\sigma} F(s-v)^{1/\sigma} \right\} ds, & \text{for } t \in [v, \bar{T}]. \end{cases}$$

Evaluate $k(t)$ at $t = \bar{T}$ and use $k(\bar{T}) = 0$

$$0 = \int_0^v \left\{ y(s) - \lambda^{-1/\sigma} B(v-s)^{1/\sigma} \right\} ds + \int_v^{\bar{T}} \left\{ y(s) - \lambda^{-1/\sigma} F(s-v)^{1/\sigma} \right\} ds$$

and then solve for λ

$$\lambda^{-1/\sigma} = \frac{\int_0^{\bar{T}} y(s) ds}{\int_0^v B(v-s)^{1/\sigma} ds + \int_v^{\bar{T}} F(s-v)^{1/\sigma} ds} = \frac{T y_1 + (\bar{T} - T) y_2}{\int_0^v B(v-s)^{1/\sigma} ds + \int_v^{\bar{T}} F(s-v)^{1/\sigma} ds}.$$

Hence, looking backward over $t \in [0, v]$ the optimal consumption path is

$$c_B^*(t|T, v) = \frac{Ty_1 + (\bar{T} - T)y_2}{\int_0^v B(v-s)^{1/\sigma} ds + \int_v^{\bar{T}} F(s-v)^{1/\sigma} ds} B(v-t)^{1/\sigma},$$

and looking forward over $t \in [v, \bar{T}]$ the optimal consumption path is

$$c_F^*(t|T, v) = \frac{Ty_1 + (\bar{T} - T)y_2}{\int_0^v B(v-s)^{1/\sigma} ds + \int_v^{\bar{T}} F(s-v)^{1/\sigma} ds} F(t-v)^{1/\sigma}.$$

2.2.2 The Outer Problem

Now that we have found the optimal consumption allocations for the past and future, conditional on a fixed retirement age T , the solution to the outer problem is given by:

$$T^*(v) = \arg \max_T \left[\begin{array}{l} \left(\int_0^v B(v-t) \frac{c_B^*(t|T, v)^{1-\sigma}}{1-\sigma} dt + \int_v^{\bar{T}} F(t-v) \frac{c_F^*(t|T, v)^{1-\sigma}}{1-\sigma} dt \right) \\ - \left(\int_0^{\min\{v, T\}} B(v-t) \psi dt + \int_v^{\max\{v, T\}} F(t-v) \psi dt \right) \end{array} \right].$$

Use the inner solutions $c_B^*(t|T, v)$ and $c_F^*(t|T, v)$ to rewrite

$$T^*(v) = \arg \max_T \left[\begin{array}{l} \frac{(\lambda^{-1/\sigma})^{1-\sigma}}{1-\sigma} \left(\int_0^v B(v-t)^{1/\sigma} dt + \int_v^{\bar{T}} F(t-v)^{1/\sigma} dt \right) \\ - \left(\int_0^{\min\{v, T\}} B(v-t) \psi dt + \int_v^{\max\{v, T\}} F(t-v) \psi dt \right) \end{array} \right],$$

and use $\lambda^{-1/\sigma}$ from the inner problem to rewrite again

$$T^*(v) = \arg \max_T \left[\begin{array}{l} \frac{(\lambda^{-1/\sigma})^{1-\sigma}}{1-\sigma} \left(\frac{Ty_1 + (\bar{T} - T)y_2}{\lambda^{-1/\sigma}} \right) \\ - \left(\int_0^{\min\{v, T\}} B(v-t) \psi dt + \int_v^{\max\{v, T\}} F(t-v) \psi dt \right) \end{array} \right]$$

or

$$T^*(v) = \arg \max_T \left[\lambda \left(\frac{Ty_1 + (\bar{T} - T)y_2}{1-\sigma} \right) - \psi \left(\int_0^{\min\{v, T\}} B(v-t) dt + \int_v^{\max\{v, T\}} F(t-v) dt \right) \right].$$

This value function is not always concave and does not always admit interior solutions so we optimize by brute force (we try every T on the computer and keep the maximizer). After obtaining $T^*(v)$, we can define the solution consumption allocation

$$c^*(t|T^*(v), v) \equiv \begin{cases} c_B^*(t|T^*(v), v), & \text{for } t \in [0, v], \\ c_F^*(t|T^*(v), v), & \text{for } t \in [v, \bar{T}]. \end{cases}$$

3 Numerical Examples

We consider standard, dynamically-consistent decisionmaking and hence $F(\tau) = e^{-\rho_F \tau}$. We also assume the individual discounts past utility exponentially, though not necessarily at the same rate that he discounts future consumption, $B(\tau) = e^{-\rho_B \tau}$. Under these assumptions, optimal consumption and retirement decisions from vantage point v can be summarized succinctly as

$$T^*(v) = \arg \max_T \left[\lambda \left(\frac{T y_1 + (\bar{T} - T) y_2}{1 - \sigma} \right) - \psi \left(\int_0^{\min\{v, T\}} e^{-\rho_B(v-t)} dt + \int_v^{\max\{v, T\}} e^{-\rho_F(t-v)} dt \right) \right],$$

$$\lambda = \left(\frac{T y_1 + (\bar{T} - T) y_2}{\int_0^v e^{-\rho_B(v-s)/\sigma} ds + \int_v^{\bar{T}} e^{-\rho_F(s-v)/\sigma} ds} \right)^{-\sigma},$$

$$c^*(t|T^*(v), v) = \frac{T^*(v) y_1 + [\bar{T} - T^*(v)] y_2}{\int_0^v e^{-\rho_B(v-s)/\sigma} ds + \int_v^{\bar{T}} e^{-\rho_F(s-v)/\sigma} ds} \times \begin{cases} e^{-\rho_B(v-t)/\sigma}, & \text{for } t \in [0, v], \\ e^{-\rho_F(t-v)/\sigma}, & \text{for } t \in [v, \bar{T}]. \end{cases}$$

Our baseline parameter values are: $\rho_F = 4\%$, $\sigma = 2$, $\bar{T} = 55$ (to reflect an economic life from ages 25 to 80), $y_1 = 1$, and $y_2 = 0$. Given these values, we set $\psi = 3.412$ to generate endogenous retirement at age 66 (model age 41). That is, $T^*(0) = 41$ when $\psi = 3.412$.

Notice that we do not need to make any assumption about the backward discount rate ρ_B when calibrating the parameter ψ because the backward rate is irrelevant to the actual consumption, saving, and retirement decisions of the individual. And, regardless of the backward rate, as the individual ages he will stick with his initial choices because he discounts future utility exponentially. However, the backward rate is relevant when computing the ideal consumption, saving, and retirement decisions of the later selves when these selves are hypothetically granted full control rights over lifetime resources. We consider three intuitive parameterizations of ρ_B :

Parameterization 1: $\rho_B = \rho_F$. Past utility is discounted the same as future utility.

Parameterization 2: $\rho_B = 0$. Past utility is not discounted.

Parameterization 3: $\rho_B = \infty$. The past provides no current utility.

Figure 1 shows the optimal retirement age $T^*(v)$ as a function of vantage point v , for all three parameterizations of ρ_B . We plot two reference lines: first, the retirement date that is optimal from the initial vantage point $T^*(0)$, which is also the retirement date that is actually experienced (because the forward looking discount function is exponential and he behaves in a dynamically consistent fashion); and second, a 45 degree line to easily observe the relation between the optimal retirement date and the current vantage point.

Notice that the individual initially wants to retire at age 66, and all three $T^*(v)$ profiles trivially equal 66 when $v = 0$ because no time has yet elapsed and therefore backward discounting is not yet relevant. All three $T^*(v)$ profiles decline early on, reflecting the individual's regret over not having saved more on the interval $[0, v]$ to enjoy an earlier retirement. The rate at which $T^*(v)$ drops is related to ρ_B in an intuitive way. The higher ρ_B , the sharper the decline. When $\rho_B = \infty$, for example, as the individual ages he experiences the sharpest regret over not having saved more to finance a very early retirement. In fact, in this case, by the time the individual reaches age 40, he wishes that he had saved enough to retire at that very moment, even though he recognizes that he is on a savings path that will lead to retirement at age 66. As he ages beyond 40, he continues to wish that he were retiring at his current vantage point (and not a moment sooner, because he wants to subject past selves to working and saving every penny of wage income).

For more reasonable assumptions such as $\rho_B = \rho_F$, the individual will eventually want to retire when he is 50, and as he ages beyond 50 he will believe it is optimal to retire earlier than his current age. Thus, there is an interesting range of ages between 50 and 66 in which this individual will be working and will recognize that he is on a dynamically consistent consumption-saving plan that leads to retirement at age 66, and yet he will wish that he were on a different plan that would have resulted in him already being retired. Unlike the case of infinite backward discounting ($\rho_B = \infty$) in which $T^*(v)$ rides the 45 degree line because those older selves always want younger selves to work, in the present case the older selves do care about the consumption and leisure experienced by their younger selves and hence do prefer that at least some of those selves could have enjoyed the utility that comes from retirement.

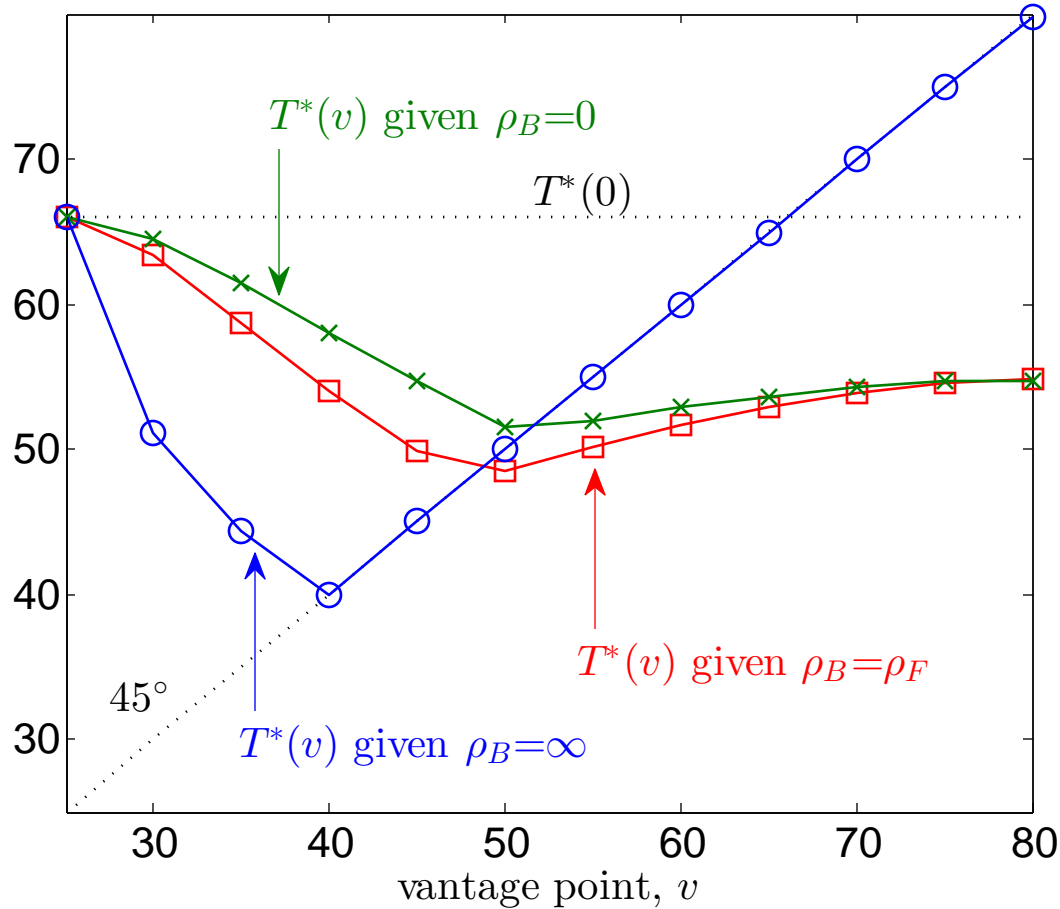
Finally, the graph of $T^*(v)$ for the case of no backward discounting ($\rho_B = 0$) is similar to the case of $\rho_B = \rho_F$. Both graphs decline until they cross the 45 degree line and then rise gently thereafter. In all cases it is clear that disagreement among the many selves about the ideal retirement date is very pronounced.

In the next set of figures we plot savings profiles. We use the following notation. A savings profile or savings allocation $k^*(t|T, v)$ is the optimal balance in the savings account at age t , condition on retirement occurring at age T and conditional on the individual's current vantage point v .

Figure 2 plots optimal savings allocations under full control rights. The allocation that is optimal from the initial vantage point $k^*(t|T^*(0), 0)$ is plotted as a reference. The other three graphs correspond to vantage point $v = 15$ (age 40), for the three different assumptions about ρ_B . This figure conveys two important points. First, we can easily see how a 40 year old would disagree with a 25 year old over how much to save for retirement, and second we can see how this disagreement depends on the particular assumptions that we make about backward discounting ρ_B .

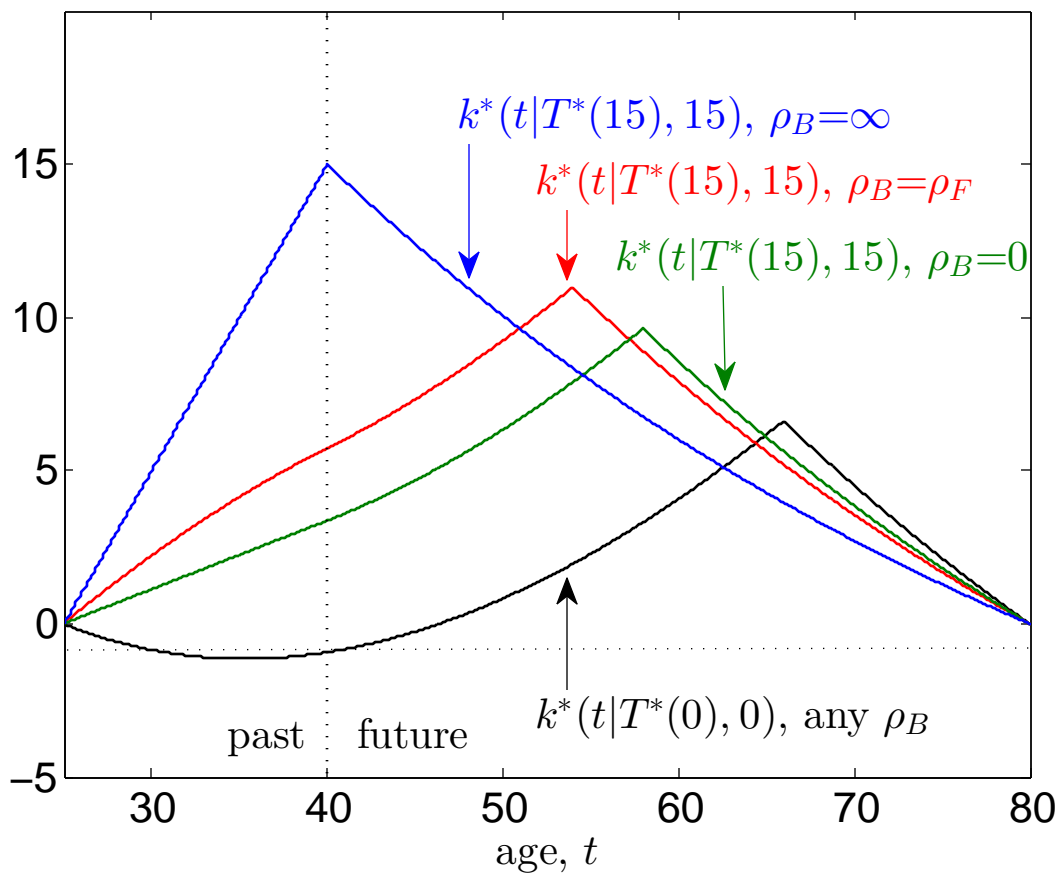
Figures 3, 4, and 5 illustrate our main point. Each of these figures decomposes the disagreement about retirement savings among the selves into the part that would occur if retirement is fixed and the overall effect when retirement is endogenous. Like Figure 2, these figures continue to use vantage point $v = 15$ (age 40) as an example to illustrate the point. The figures differ only in the value that is assigned to ρ_B . The point of these figures is to show how the disagreement over how much should be in the savings account at the date of retirement is amplified by the presence of the retirement margin. Consider the case of $\rho_B = \rho_F$ (Figure 3). If we hold retirement fixed at age 66, then a 40 year old would wish that he was on a path that would ultimately involve 11% more savings by age 66 than the amount that will actually be realized. Alternatively, under endogenous retirement, a 40 year old would wish that he was on a very different path that would ultimately involve 66% more savings for retirement (given the earlier retirement age, 54, that is optimal from his current perspective), compared to the level of savings at the age of retirement that he will actually experience. Age 40 is just one possible vantage point to show this result. But the point is that the disagreement among the various time-dated selves over how much to save becomes severely amplified when the retirement mechanism is operative in the model.

Figure 1. Optimal Retirement Age under Full Control Rights



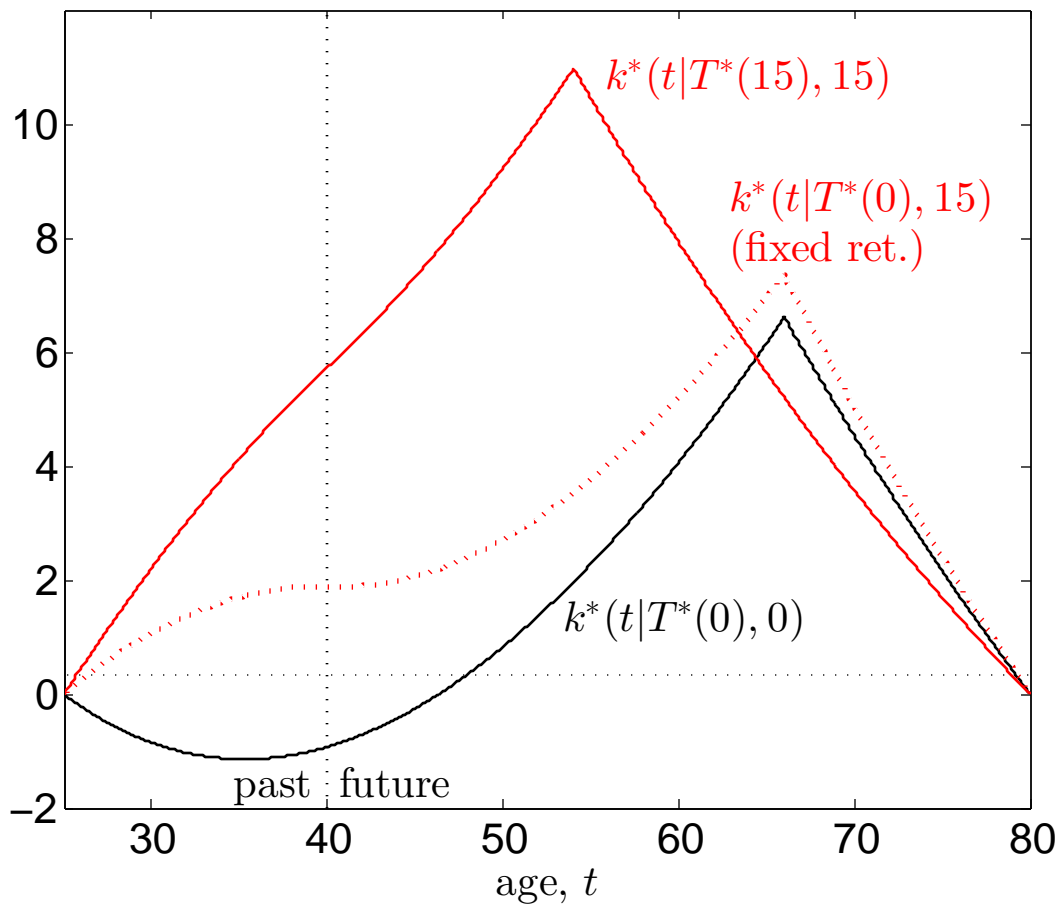
The forward discount rate, ρ_F , is set to 4%.

Figure 2. Optimal Savings under Full Control Rights



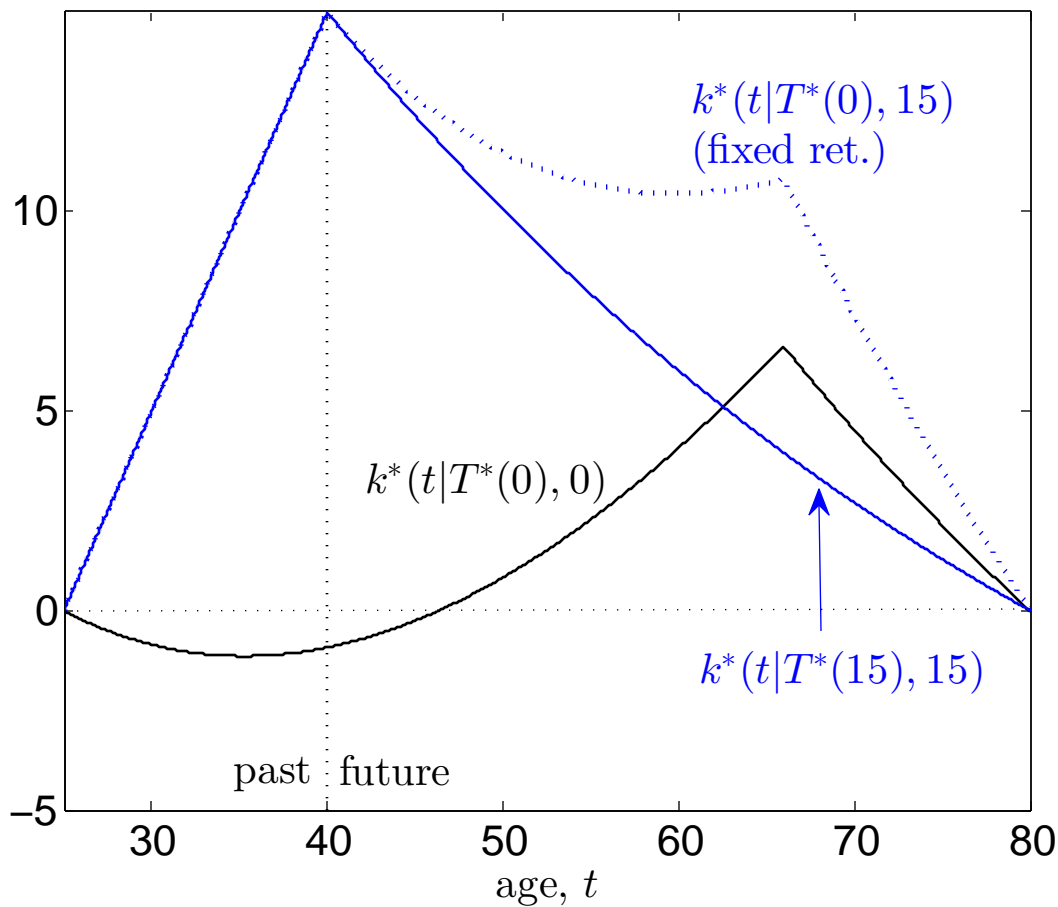
The forward discount rate, ρ_F , is set to 4%, and the vantage point is $v = 15$ (age 40).

Figure 3. Savings Decomposition with $\rho_B = \rho_F$: The Role of Retirement



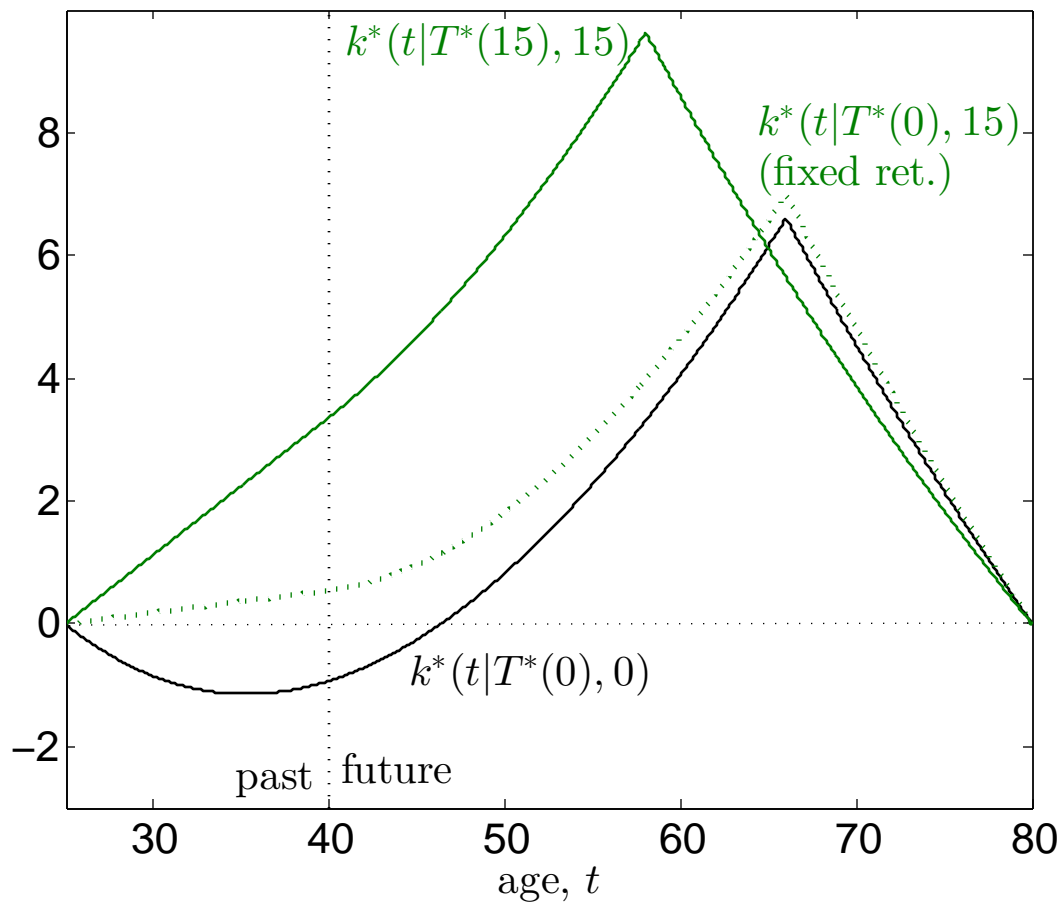
The forward discount rate, ρ_F , is set to 4%, and the vantage point is $v = 15$ (age 40).

Figure 4. Savings Decomposition with $\rho_B = \infty$: The Role of Retirement



The forward discount rate, ρ_F , is set to 4%, and the vantage point is $v = 15$ (age 40).

Figure 5. Savings Decomposition with $\rho_B=0$: The Role of Retirement



The forward discount rate, ρ_F , is set to 4%, and the vantage point is $v = 15$ (age 40).