Robust Efficient Decision Rules

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Abstract

I define robust efficient decision rules as those that are not Pareto dominated within the set of decision rules that satisfy robust feasibility – incentive compatibility and other constraints. Examples are provided in public and private good environments, where I show non-existence of classically efficient decision rules and then construct robust efficient decision rules. Robust efficiency is characterized as *ex ante* and *interim* incentive efficiency when individuals have rich type spaces. Robust efficiency is also characterized by non-existence of a robust common knowledge event, appropriately defined, whereby the individuals would agree to replace a given feasible decision rule with another one. Anonymous decision rules are discussed to facilitate the examples.

Keywords: Efficiency, Robustness, Pareto dominance, Incentive compatibility, Private Goods, Public Goods

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1 Introduction

Efficiency is central to economics. When all the information in an economy is known by all individuals, classical (or Pareto) efficiency due to Pareto (1896) has been the indisputable fundamental definition of an efficient allocation. Such classical conditions of complete information obtain when various pieces of relevant information, initially held by different individuals, have all been aggregated through a communication system, henceforth *decision rule*. For example; in an exchange economy, if the individuals can credibly communicate their demands, market clearing prices may aggregate all relevant information and a classically efficient allocation obtains.[1] Notably, classical efficiency is robust in that it does not depend on the details of the specification of the individuals' private information. A plethora of examples have shown, however, that it is often impossible to attain a classically efficient allocation and provide the individuals with the incentives to truthfully reveal their private information in an equilibrium outcome of any desirable economic institution.[2]

In his seminal work, Harsanyi (1967-68) described Bayesian environments with incomplete information where the individuals hold (possibly subjective) prior beliefs over uncertain parameters. This paved the way for the study of incentive problems and called for a definition of efficiency at different stages of revelation of information, and production and allocation decisions. Wilson (1968) provided such a definition without considering the incentive constraints,

[1]A sufficient condition for The First Welfare Theorem is that the number of individuals is large and there are no consumption externalities, see Makowski and Ostroy (1995) and Ostroy and Segal (2012). Gul and Postlewaite (1992) construct Bayesian incentive compatible decision rules that converge to the Walrasian allocation as the number of individuals becomes large, and McAfee (1992b) constructs an example of such dominant-strategy incentive compatible double auction. For recent work on competitive equilibrium and classical efficiency see Richter and Rubinstein (2015).

[2]Myerson and Satterthwaite (1983) proved the impossibility of classically efficient bilateral trade. Gibbard (1973) and Satterthwaite (1975) proved that under the full domain, any decision rule that is classically efficient and dominant-strategy incentive compatible or strategy-proof must be dictatorial and Reny (2001) showed the close relationship to the theorem by Arrow (1963). In exchange economies with a finite number of individuals, Hurwicz (1972) showed that the Walrasian correspondence is not strategy proof, Palfrey and Srivastava (1987) showed it is not Bayesian incentive compatible, and Barberà and Jackson (1995) showed that it is not strategy proof when the number of individuals becomes large. See also sections 3 and 4 here.

and Holmström and Myerson (1983), in yet another seminal contribution, provided the definition of *ex ante* and *interim* incentive efficiency for Bayesian environments with incomplete information.[3] Recently, a lively discussion has been rekindled regarding environments where the allocation rules and the individuals' incentives are robust to the details of the specification of the individuals' information.[4] While robust notions of incentive compatibility are relatively well understood, there has been no similarly appealing general definition of robust incentive efficiency and most of the literature has focused on characterizing conditions under which robust classically efficient decision rules exist.[5]

In this paper I define robust efficiency. Robust efficient decision rules are those that are not Pareto dominated within the set of feasible decision rules. Feasibility constraints are given by a robust notion of equilibrium, i.e., incentive compatibility, and other constraints relevant to the economic environment at hand. To illustrate the notion, I study robust efficient decision rules in two important environments where classically efficient decision rules do not exist, one with public goods, and the other with private goods. The definition of robust efficiency, Definition 2 in Section 2, is elementary and whenever the feasibility constraints are not binding, it coincides with classical efficiency, as it properly ought to. Thus, on the one hand, the notion extends classical efficiency to environments where classically efficient decision rules do not exist. On the other hand, robust efficiency extends the notions of *ex ante* and *interim* incentive efficiency to robust environments. In Section 5, following the approach of Bergemann and Morris (2005), I make this relation explicit by showing that robust efficient decision rules can

[3]Gilboa et al. (2014) argue that classical efficiency is too strong and propose a domination of allocations, based on existence of a shared belief whereby the individuals could effect a mutually-beneficial bet.

[4]Mertens and Zamir (1985) and Brandenburger and Dekel (1993) provided a formal definition of rich type spaces and Bergemann and Morris (2005) provided a mechanism-design framework to study various robust notions of incentive compatibility; in particular *ex post* incentive compatibility and *strategy-proofness*. See also Ledyard (1978) and Chung and Ely (2007).

[5]Barberà et al. (2010) and Barberà et al. (2016) characterize conditions under which strategy-proofness implies coalitional-strategy-proof decision rules; strategy-proofness with respect to the grand coalition is classical efficiency. A recent example of a strategy-proof classically-efficient auction can be found in Milgrom and Segal (2015).

be thought of as *ex ante* or *interim* incentive efficient under the individuals' sufficiently rich type spaces.

Holmström and Myerson (1983) discussed the notion of *ex post* incentive efficiency which they defined as constrained Pareto efficiency under the feasibility constraints of Bayesian incentive compatibility. They viewed the *ex post* stage as one where all the information has become publicly available. They therefore discarded *ex post* incentive efficiency on the grounds that at the *ex post* stage, there are no longer any informational or incentive problems and one thus ought to consider classical efficiency as the corresponding efficiency concept. First, as discussed above, in many environments there are issues with the existence of classically efficient decision rules. Second, if a decision rule is not feasible, then it cannot constitute a viable alternative to a feasible economic institution. For example, a classically efficient decision rule may well dominate a feasible decision rule, however, if it is not feasible, then the former does not constitute a more efficient alternative to the latter. In particular, if a decision rule is not incentive compatible, then it cannot describe an equilibrium outcome of any economic institution so that it is not feasible. This is the contrapositive of the revelation principle – see Corollary 6 in Section 6.1 where I provide the statement for the notion of *ex post* Nash equilibrium, or *ex post* incentive compatibility. Thus, in a general game form, rather than a description of a stage in the revelation of information, the *ex post* stage is a description of a stage at which given constraints must hold, and the corresponding incentive constraints on decision rules are equivalent to robust equilibrium constraints.[6] In environments where classically efficient decision rules do not exist, and also more generally, robust efficiency provides a tool to study economic

[6]This is also suggested by Ledyard (1978) and Bergemann and Morris (2005), and it applies to the *interim* stage in its relation to Bayes-Nash equilibrium as well. At any rate, this holds for any equilibrium notion satisfying the revelation principle; even more restrictive equilibrium constraints apply to equilibrium refinements, in particular, refinements used in dynamic settings. For more on the revelation principle in a Bayesian setting see especially Myerson (1991). Note also that the later the stage, the stronger the corresponding constraints, which implies that robust efficiency, which corresponds to the *ex post* stage, is weaker than the *interim* notion or the *ex ante* notions of constrained efficiency.

institutions that are both feasible and as efficient as possible.

I illustrate robust efficiency in two examples in sections 3 and 4. The first example is a classical problem of providing a public good. There I characterize the robust efficient decision rules within the set of decision rules that satisfy incentive compatibility, participation, no subsidies, and anonymity constraints; as a corollary, I show the impossibility of classically efficient decision rules. The second example is in a private-good environment with no consumption externalities, where a decision must be made as to which individual to allocate a good.[7] I first show non-existence of classically efficient decisions rules, which is novel, and then construct a decision rule that is robust efficient and anonymous. A common feature of the robust efficient decision rules in these two examples is that an allocation that does not attain a maximal surplus is implemented with a positive probability.[8]

One question is whether and in what sense robust efficient decision rules are durable. To address this question, it is necessary to provide a whole separate framework, which lies outside the scope of the present analysis; I provide some preliminary discussion in Section 6. There I first define events that are robust common knowledge, Definition 8. Next I provide a characterization of robust efficient decision rules by non-existence of such a robust common knowledge event, wherein a given decision rule could be dominated, Theorem 7. Finally, to facilitate the aforementioned examples, I pay special attention to symmetric environments and anonymous decision rules, Theorem 8 and preceding definitions. Anonymity *per se* is a sensible desideratum in many environments, especially markets.

In the tradition of Pareto efficiency, robust efficiency is impartial to questions of redistribution. Maximizing any social welfare criterion will result in a robust efficient decision

[7]This example is closely related to the Solomon's problem and the problem of dissolving a partnership – see Section 4 for details and references.

[8]Robust efficiency may also be attained by burning some numeraire: to my knowledge, Copic (2017) provides the first such example in the context of bilateral trade, which is complementary to the partial characterization in Copic and Ponsatí Obiols (2016).

rule under the given constraints, Theorem 1.[9] Characterizing robust efficient decision rules is analytically tractable so that robust efficiency is a useful starting point for normative considerations. Robust efficiency provides a conceptual framework and an analytical tool for normative analysis in robust environments.

To summarize, this paper is structured as follows. In Section 2, I define robust efficiency and discuss convex and closed feasibility constraints, such as incentive compatibility, individual rationality and no subsidies. In Sections 3 and 4 respectively, I give examples in public and private-good environments. In Section 5, I show the equivalence between robust efficiency and *ex ante* or *interim* constrained efficiency in rich type spaces. In Section 6, I recall the revelation principle, consider durability of robust efficient decision rules, discuss anonymous decision rules, and give concluding remarks.

2 Robust efficiency and simple implications

An environment is defined by $(N, \boldsymbol{\Theta}, Y, c, u)$. The set $N = \{1, ..., n\}$ describes the individuals in the economy. For each individual $i \in N$ the set Θ_i describes the finite set of i's types, and $\boldsymbol{\Theta} = \Theta_1 \times ... \times \Theta_n$ thus describes all the informational states of the economy;[10] for any set X, $\bar{X}$ denotes the set of probability measures over X, e.g., $\bar{\boldsymbol{\Theta}}$ is the set of probability measures over $\boldsymbol{\Theta}$. The set Y describes the set of (deterministic) feasible allocations. I will for the most part assume that Y is given by a finite number of alternatives $y_0 \in Y_0$ and the vectors of transfers y between the individuals, $y_N \in Y_1 \times ... \times Y_n$, $Y_i = \mathbb{R}$, where $y_i > 0$ when i receives a payment, and $y_i < 0$ when i makes a payment. An allocation $y \in Y$ is thus given by $y = (y_0, y_N)$. Most

[9] In various robust settings, examples have been provided of maximizing different social welfare criteria, such as regret-free or maximin, see Bergemann and Schlag (2008), Bergemann and Schlag (2011), Carrasco and Moreira (2013), and Carrasco et al. (2015).

[10] Boldface notation $\boldsymbol{\Theta}$ is used to denote the entire parameter space, so as not to cause any confusion in symmetric environments defined below, where Θ is used to denote an individual's set of private parameters, i.e., $\Theta_i = \Theta$.

of the definitions here extend easily to environments with no transfers by setting $Y \equiv Y_0$. The function $c : Y_0 \to \mathbb{R}$ describes the production costs (in terms of the numeraire commodity) associated with different allocations, where $c(y_0)$ is the cost of allocation $y_0 \in Y_0$, and I assume that $c(y_0) < \infty, \forall y_0 \in Y_0$. Whenever $c \equiv 0$, c can be omitted from the description of the environment.[11]

For each $\theta \in \boldsymbol{\Theta}$, individual i's preferences are described by a von Neumann-Morgenstern utility function $u_i(., \theta) : \bar{Y} \to \mathbb{R}$, where $u_i(\bar{y}, \theta)$ is linear in $\bar{y}$, since $\bar{y}$ is a probability measure; thus $u = (u_1, ..., u_n)$ describes the individuals' utility functions. The parameter θ_i incorporates all of i's payoff-relevant private information, θ_i is not known by other individuals, and the utility functions $u_i(., .)$ are common knowledge. I assume that u_i is increasing in the transfer y_i, for each $y \in Y$, and each $\theta \in \boldsymbol{\Theta}$. In the examples, I make the more restrictive assumption of quasi-linearity; that is, the utility functions are additively separable in the allocation and the monetary transfer, quasi-linear with respect to transfers, and are normalized so that 1 unit of money is worth 1 unit of utility to all agents, $u_i(y, \theta) = y_i + \nu_i(y_0, \theta)$.

For the purpose of this section, it is common knowledge that the individuals' types lie in the parameter space $\boldsymbol{\Theta}$. The individuals have no further information regarding the details of the descriptive statistics of these types, in particular, they do not know the probability distribution over types.[12] A decision rule (or a social choice function) is a mapping $d : \boldsymbol{\Theta} \to \bar{Y}$. I denote by $\mathcal{D}$ the set of all decision rules such that $\sum_{i \in N} d_i(\theta) \le K$, $\forall \theta \in \boldsymbol{\Theta}$, for all $d \in \mathcal{D}$ and some

[11] For example, N may be a set of sellers and buyers of some number of objects. In this case, Y is the set of possible allocations of the objects and the vector of transfers between the individuals, and $c(y_0) = 0, \forall y_0 \in Y_0$. When an allocation y_0 involves some production, then it may be that $c(y_0) > 0$, for example, in the problem of providing a public good - see Section 3 below.

[12] In the social choice literature, such a specification of an environment with incomplete information is standard. For example, in an environment with private values, where i's preferences are affected only by his own private parameter θ_i, present description defines a (restricted) domain of i's preferences, $\{u_i(., \theta_i), \theta_i \in \Theta_i\}$; equilibrium (or other solution) concepts that are considered, do not depend on the probability distribution over individuals' types, e.g., strategy-proofness. Alternatively, one can imagine a Bayesian approach to incomplete information conceived by Harsanyi (1967-68), and then require that the model be immune to the changes of the structure of the environment – for example, robust to all possible individuals' priors as in Ledyard (1978). I refer to Section 5 for such a description in the framework of rich type spaces as in Bergemann and Morris (2005).

constant $K < \infty$, in particular, I prohibit decision rules that require infinite subsidies.

Key to the definition of classical efficiency is the Pareto dominance relation.

Definition 1. *A decision rule d' Pareto (or ex post) dominates d, denoted by $d' \succ d$, if,*

$$u_i(d'(\theta), \theta) \geq u_i(d(\theta), \theta), \quad \forall \theta \in \mathbf{\Theta}, \ \forall i \in N, \tag{1}$$

with at least one strict inequality. A decision rule $d \in \mathcal{D}$ is classically efficient *(or Pareto efficient or ex post efficient), if there does not exist a $d' \in \mathcal{D}$, such that d' Pareto dominates d.*

The Pareto dominance relation and the notion of classical efficiency are both robust: Pareto dominance relation depends on the parameter domain $\mathbf{\Theta}$ and the payoff information, but is independent of further details of the specification of the individuals' information.

If in a given environment there are no constraints on the set of feasible decision rules, then classical efficiency is easily attainable. For example, in any quasi-linear environment with transfers and no constraints, a decision rule is efficient as long as $d_0(\theta) = \max_{y_0 \in Y_0} \sum_i \nu_i(y_0, \theta)$. However, in most any economic environment the relevant decision rules must satisfy some constraints pertinent to the informational and contractual aspects of the allocation problem. The decision rules which do not satisfy those constraints are infeasible, for one reason or another. The set of feasible decision rules is given by some $D \subset \mathcal{D}$ and classically efficient decision rules may or may not be feasible. Nevertheless, there may still be rules which are more efficient than others in the sense of Pareto domination. This suggests a natural way to extend the definition of classical efficiency to general robust environments, in particular, to those where classically efficient decision rules are infeasible. The following is the central notion of this paper.

Definition 2. *Given a $D \subset \mathcal{D}$, a decision rule $d \in D$ is robust efficient in D if there does not exist a $d' \in D$, such that d' Pareto dominates d.*

Denote by $D^* \subset D$ the set of robust efficient decision rules.[13] If there are no constraints, then the set $\mathcal{D}^*$ of robust efficient decision rules in $\mathcal{D}$ is the set of classically efficient decision rules.

A different approach to defining constrained efficient decision rules is by maximizing a social welfare criterion where different individuals, or indeed even different individuals' types, may be assigned different weights. When considering *ex post* constrained efficient decision rules, the appropriate measurability requirement is that the weights in the social welfare function be measurable with respect to the individuals' types, see Holmström and Myerson (1983) and Wilson (1968). Given a $D \subset \mathcal{D}$, a decision rule $d \in D$ is a robust social welfare maximizer if,

$$\begin{gathered} \exists \lambda : \boldsymbol{\Theta} \to R^N_{++}, \text{ s.t.,} \\ d \in \arg\max_{d' \in D} \sum_{\theta \in \boldsymbol{\Theta}} \sum_{i \in N} \lambda_i(\theta) u_i(d(\theta), \theta). \end{gathered} \tag{2}$$

Denote by $D^{\mathcal{W}} \subset D$ the set of robust social welfare maximizers. The next theorem is a straightforward extension of well-known standard results, see Wilson (1968) and especially Holmström and Myerson (1983). Denote the closure of a set O by $cl(O)$.[14]

Theorem 1. *Let $D \subset \mathcal{D}$ be convex and closed. Then $cl(D^{\mathcal{W}}) = D^*$.*

Most economic constraints are convex and closed. Before discussing the examples, I enumerate some of these constraints, all of which are standard. I refer to Bergemann and Morris (2005) for a more detailed discussion and further results, especially concerning the incentive compatibility constraints in robust settings.

[13] Such decision rules are indeed robust *constrained* efficient: the constraints are specified by D so that no confusion should arise with the notion of unconstrained, i.e., classical, efficiency; As argued below, incentive constraints should necessarily be included in the specification of D, however, using "robust incentive efficient" may be misleading since in many environments there are additional natural constraints, e.g., individual rationality and market clearing (free disposal) constraints in market environments.

[14] In the proof of the following theorem it is useful to observe that, as far as Pareto dominance is concerned, we can limit the attention to the space of the individuals' utilities endowed with the Euclidean metric and the corresponding topology. Note that in a given decision rule, the individuals' utilities are specified by $n \times |\boldsymbol{\Theta}|$ points in the Euclidean space.

A key constraint is incentive compatibility. When a decision rule satisfies incentive compatibility, it is usually called a direct revelation mechanism, or simply a mechanism. Two standard notions of incentive compatibility suitable for robust environments are the dominant-strategy incentive compatibility and the somewhat weaker *ex post* incentive compatibility.

Definition 3. *A decision rule d is dominant-strategy incentive compatible (*strategy-proof*) if,*

$$u_i(d(\theta_i, \theta'_{-i}), \theta) \geq u_i(d(\theta'_i, \theta'_{-i}), \theta), \forall \theta_i, \theta'_i, \theta_{-i}, \theta'_{-i}.$$

Definition 4. *A decision rule d is (ex post) incentive compatible, if,*

$$u_i(d(\theta_i, \theta_{-i}), \theta) \geq u_i(d(\theta'_i, \theta_{-i}), \theta), \forall \theta_i, \theta'_i, \theta_{-i}.$$

From now on, by incentive compatibility I mean the weaker *ex post* incentive compatibility, however, in the two main examples of sections 3 and 4 that will not play a role: In private-values environments it is immediate that the above two notions of incentive compatibility are equivalent. For the sake of completeness we state this well-known fact here as a proposition.

Proposition 1. *In a private-values environment where $u_i(y, \theta) = u_i(y, \theta_i), \forall i, y, \theta$, a decision rule is ex post incentive compatible if and only if it is* strategy-proof.

Proof. Let a decision rule d be *ex post* incentive compatible, then,

$$u_i(d(\theta), \theta) = u_i(d(\theta), \theta_i) \geq u_i(d(\theta'_i, \theta_{-i}), \theta_i) = u_i(d(\theta'_i, \theta_{-i}), \theta_i, \theta'_{-i}), \forall \theta_i, \theta'_i, \theta_{-i}, \theta'_{-i},$$

so that d is strategy-proof. The other direction is evident. □

Other standard constraints that may be imposed on decision rules are individual rationality and budget balance (or no subsidies). If each individual can under any circumstances obtain a base level of utility, usually normalized to 0, then any feasible mechanism must satisfy individual rationality, that is,[15]

$$u_i(d(\theta), \theta) \geq 0, \forall \theta, \forall i.$$

Budget balance is a reasonable requirement for environments when there is no external agency willing to provide subsidies to the system. Then, in every state θ, the sum of the transfers net the cost of the allocation must be non-positive (or zero, in the case of exact budget balance). A decision rule d satisfies (*ex post*) budget balance, if,

$$\sum_{i=1}^{n} d_i(\theta) \leq -c\left(d_0(\theta)\right), \forall \theta,$$

and it satisfies exact budget balance if the above is an equality.[16]

In some environments additional constraints may be required, for example, the *market clearing* constraints when Y_0 can be interpreted as a market – that is, the markets ought to clear for goods other than the numeraire.

The above constraints will define some subset of decision rules $D \subset \mathcal{D}$. These constraints – incentive compatibility, individual rationality, budget balance, market clearing, and other constraints – assure that D is a closed and convex set so that Theorem 1 applies. An important example of non-convex constraints are those derived from voting correspondences, specifically, when individual rationality constraints must be satisfied only by a simple majority of the individuals.

[15]For example, there may exist some $\tilde{y}^i \in Y_0$ such that when y is given by $y_0 = \tilde{y}^i$ and $y_{N,i} = 0$ then,

$$u_i(y, \theta) = 0, \forall \theta$$

In addition, for each i there may exist a type $\underline{\theta}_i$, which allows i to opt out, that is, when i reports that type, the allocation is $\tilde{y}_i \in Y_0$. Generally, there is an allocation $y_0 \in Y_0$ such that when the transfer to individual i is 0, she obtains a zero utility.

[16]Recall that a decision rule is a mapping into lotteries over allocations. Therefore, individual rationality and budget balance as specified here may, for each draw of types, be interpreted either in terms of the expectation of that lottery, or as point-wise constraints for each realization of that lottery, whichever may be more appropriate for the application at hand.

3 Example 1: Providing a public good

The first illustrative example is a classical problem where N individuals must decide whether or not to finance a non-excludable, non-rival, indivisible public good. Thus, $Y_0 = \{0, 1\}$, where $y_0 = 1$ denotes the outcome whereby the public good is provided and $y_0 = 0$ the outcome whereby it is not provided. Each individual privately values the good at $\theta_i \in \Theta = \{v^L, v^H\}$, $0 < v^L < v^H < 1$, so that $\boldsymbol{\Theta} = \Theta^n$; the cost of the public good is normalized to 1 so that $c(0) = 0$ and $c(1) = 1$. The individuals have quasilinear utility functions, $u_i(y, \theta_i) = y_i + y_0\theta_i$, i.e., $\nu_i(y_0, \theta) = y_0\theta_i$. I further assume that,

$$(N-1)v^H + v^L > 1, \tag{3}$$

$$(N-2)v^H + 2v^L < 1. \tag{4}$$

In a classically efficient allocation, the public good should be provided if either $N-1$ or all individuals have a high value by (3); Otherwise the good should not be provided by (4). When the individuals' reports of their values are θ, denote by $\varphi(\theta)$ the probability that the public good is provided. The transfer $y_i(\theta)$ is interpreted as the tax paid by the individual i.

The set of feasible mechanisms is thus given by direct revelation mechanisms, requiring no external subsidies and where no individual can be forced to pay for the public good. I will impose an additional *no private benefits* constraint, or simply, no benefits. No benefits requires that no individual can directly benefit by receiving a monetary payment from the mechanism, that is, $y_i(\theta) \geq 0, \forall\theta, \forall i$. Every classically efficient decision rule ought to satisfy budget balance with equality. For that reason, and to avoid some complications, I further strengthen the budget balance requirement to exact budget balance.[17] The feasible set D is thus given by decision

[17] Kuzmics and Steg (2017), argue that it is desirable to satisfy budget balance with equality for normative reasons. In settings with continuous type spaces and no budget breaker, Copic (2017) demonstrates that the usual reduced-form representation of incentive compatible mechanisms due to Myerson (1981) is without loss of generality only with strict budget balance.

rules satisfying incentive compatibility, individual rationality, budget balance and no benefits.

In the present environment, it has been shown under various circumstances that no classically efficient allocation rules exist. After initial impossibility results by Arrow (1963), Gibbard (1973), and Satterthwaite (1975) in ordinal environments in unrestricted domains, where each individual can hold any preference over the set of possible social allocations, Roberts (1976) showed that when the number of individuals is very large, the Lindahl equilibrium allocation rule, which is classically efficient and individually rational, is not even approximately incentive compatible. Walker (1980), Zhou (1991), Barberà and Jackson (1994), and Serizawa (1999) showed non-existence of strategy-proof, efficient, and anonymous decision rules, and Ostroy and Segal (2012) showed non-existence with any finite number of individuals. With the impossibility results by D'Aspremont and Gerard-Varet (1979) and Arrow (1979), a separate strain of literature addressed the *interim* (Bayesian) incentive compatibility. Guth and Hellwig (1986) studied expected welfare-maximizing public-good provision mechanisms, Mailath and Postlewaite (1990) showed that as the number of individuals tends to infinity, the probability of providing the public good tends to zero, and Ledyard and Palfrey (2007) characterized the *interim* incentive efficient mechanisms.

Consequently, the literature has evolved along two different lines: to characterize decision rules that satisfy certain additional assumptions, such as anonymity;[18] and to describe decision rules that are as efficient as possible in some set.[19] Here I show non-existence of classically efficient decision rules in the simplest possible public good environment and then characterize the set of anonymous robust efficient decision rules. The characterization yields a

[18]Moulin (1994) considers serial cost-sharing anonymous mechanisms for excludable public goods, Serizawa (1999) characterizes symmetric mechanisms for divisible public goods with convex costs as ascending-auction like mechanisms, Bierbauer and Hellwig (2016) characterize symmetric coalition-proof mechanisms, and Kuzmics and Steg (2017) characterize deterministic mechanisms.

[19]Ledyard (2006) provides a review of public-good provision mechanisms and argues for comparisons on the grounds of efficiency of outcomes, as well as transparency of the mechanisms themselves. Smith (2010) shows that as long as each individual might be willing to pay for the public good by herself, there are robust decision rules that under some beliefs do better than strategy-proof voting rules.

clear illustration of efficiency losses associated with the above constraints, via the probability that the public good is not provided when it should be. I begin by discussing a well-known class of incentive compatible mechanisms.

Vickrey-Clarke-Groves (VCG) mechanisms.

A common class of mechanisms to consider are the Vickrey-Clarke-Groves (VCG) schemes, see Vickrey (1961), Clarke (1971) and Groves (1973). A VCG scheme is a demand revealing direct revelation mechanism which attains the allocation $y_0 \in Y_0$ that maximizes the aggregate surplus. In this case, $\varphi \in \{0, 1\}$, and $\varphi(\theta) = 1$, if and only if, $\sum_{i=1}^{N} \theta_i \geq 1$.

In particular, a VCG scheme with a fixed-size public good reduces to the Clarke *pivot* rule, see Clarke (1971) and Green and Laffont (1977), where an individual i is pivotal if the allocation changes when she changes her report. We have two possibilities,

1. $(N-2)v^H + v^L - \frac{N-1}{N} \geq 0$.

   Then,

$$y_i(\theta) = \begin{cases} \frac{1}{N}, & \text{if } \sum_{i=1}^{N} \theta_i \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

2. $(N-2)v^H + v^L - \frac{N-1}{N} < 0$.

   In this case observe that $(N-3)v^H + 2v^L - \frac{N-1}{N} \geq 0$, by (3) and (4). Then, $y_i(\theta) = \frac{1}{N}$, if $\theta_i = v^H, \forall i$, and $y_i(\theta) = 0$, if there are at least 2 individuals with a low value, i.e., $\theta \in \mathbf{\Theta}$, s.t., $|\{\theta_i = v^L\}| \geq 2$. When θ is such that $|\{\theta_i = v^L\}| = 1$, then by efficiency of the social allocation y_0, $\varphi(\theta) = 1$, and

$$y_i(\theta) = \begin{cases} \frac{1}{N}, & \text{if } \theta_i = v^L, \\ 1 - \left((N-2)v^H + v^L\right), & \text{if } \theta_i = v^H. \end{cases}$$

In both of these possibilities, it is immediate to verify incentive compatibility. However, in the first possibility, the mechanism fails individual rationality of the individual with a low value when the public good is provided (since $v^L < \frac{1}{N}$, by (3) and (4)). In the second possibility, the mechanism does not fail individual rationality, but when one individual has a low value, the total payments exceed the cost of the public good. The reason is that $y_i(v^H) = 1 - \big((N-2)v^H + v^L\big) > \frac{1}{N}$, so that when θ is such that $|\{\theta_i = v^L\}| = 1$ we obtain,

$$\sum_{i=1}^{N} y_i(\theta) = y_i(v^L) + (N-1)y_i(v^H) = \frac{1}{N} + (N-1)y_i(v^H).$$

While the allocation y_0 is efficient, the mechanism is not classically efficient as some numeraire has to be disposed of. Therefore, in this environment, a VCG scheme is neither classically efficient nor does it belong to the feasible set D.

I now consider the set of anonymous mechanisms in D; denote this set by $\bigcirc \subset D$. In an anonymous decision rule, when the names of individuals are permuted, so are their consumption bundles (see formal definitions 10, 11, and 12 in Section 6). Such a restriction seems sensible, given that all individuals are *a priori* (that is, *ex ante*) identical so that it seems desirable that they be treated equally when determining the decision rule to be used (see also Section 5). The set $\bigcirc$ is non-empty, for example, the mechanism where the public good is never provided and no individual makes any payment satisfies all the requirements, however, this mechanism is evidently not particularly efficient and the question is whether more efficient mechanisms in $\bigcirc$ exist. In summary, for the purpose of the example, the set $\bigcirc$ of feasible mechanisms is given by incentive compatibility, individual rationality, strict budget balance, no private payments, and anonymity. I characterize the robust efficient mechanisms $\bigcirc^* \subset \bigcirc$.

Incentive compatibility will bind for the individuals to not misrepresent their values downward. Hence, all robust efficient mechanisms will have the property that the public good is provided with certainty when $\theta_i = v^H, \forall i$. Then, by symmetry, all individuals will pay the same tax, where the sum of these taxes is the cost of the good, so that $y_i = \frac{1}{N} < v^H$. Whenever more than one individual has a low value, individual rationality and budget balance imply that the public good is not provided and all taxes are 0.

The crucial case is when exactly one individual has a low value, that is $\theta \in \boldsymbol{\Theta}$ is such that, $|\{i \mid \theta_i = v^L\}| = 1$. The good will then be provided with some probability, denoted by α. Conditional on the good being provided, by symmetry, all individuals with a high value will pay the same tax, denoted by y^H; denote by y^L the tax paid by the low value individual.

By above, we can limit the attention to mechanisms specified by the triplet (α, y^H, y^L). For mechanisms in $\bigcirc$ these quantities satisfy the following constraints,

$$v^H - \frac{1}{N} \geq \alpha(v^H - y^L), \tag{5}$$

$$(N-1)y^H + y^L = 1, \tag{6}$$

$$v^H \geq y^H \geq 0, \tag{7}$$

$$v^L \geq y^L \geq 0. \tag{8}$$

(5) is the incentive constraint of high value individuals; for a low value individual, her incentive constraint is automatically satisfied since $v^L < \frac{1}{N}$. Equation (6) is the strict budget balance constraint, and (7) and (8) are the individual rationality and no benefits constraints of the high and the low value individuals, respectively.

For $\lambda \in [0, 1]$, define,

$$\begin{aligned}
\nu^H(\lambda) &= \lambda v^H + (1-\lambda)\frac{(1-v^L)}{N-1}, \\
\nu^L(\lambda) &= \lambda(1-(N-1)v^H) + (1-\lambda)v^L, \\
\alpha(\lambda) &= \frac{v^H - \frac{1}{N}}{v^H - v^L + \lambda((N-1)v^H + v^L - 1)},
\end{aligned}$$

and define by d_λ the decision rule given by $(\alpha(\lambda), \nu^H(\lambda), \nu^L(\lambda))$. The next proposition characterizes the set of robust efficient public good provision mechanisms $\bigcirc^*$ in this environment.

Proposition 2. *The set $\bigcirc^*$ is given by the convex hull of the set $\{d_\lambda \mid \lambda \in [0,1]\}$.*

Proof. The key event to consider is when all individuals except perhaps one have a high value of the public good. There are two cases when the value of the good to that individual is either y^L or y^H. Given y^L, the probability of providing the public good must be maximal, so that (5) must hold with equality, that is,

$$\alpha = \frac{v^H - \frac{1}{N}}{v^H - y^L}. \tag{9}$$

Similarly, (6) must hold with equality – otherwise taxes to some individuals may be reduced without reducing the probability of providing the good. Thus,

$$(N-1)y^H + y^L = 1.$$

In order to satisfy (7) and (8) the highest possible tax to the low value individual is v^L and the lowest is $1-(N-1)v^H$ so that admissible taxes to the low value individual are given by,

$$\nu^L(\lambda) = \lambda v^L + (1-\lambda)(1-(N-1)v^H), \lambda \in [0,1]$$

By exact budget balance admissible taxes to the high value individuals are given by $\nu^H(\lambda) = \frac{1-\nu^L(\lambda)}{N-1}$, and by substituting the expression for $\nu^L(\lambda)$ in place of y^L in (14) we obtain the expression for $\alpha(\lambda)$. By (3) and (4), $\alpha(\lambda) \in (0,1)$, for $\lambda \in (0,1)$ and by construction, $d_\lambda \in D$, $\forall \lambda \in [0,1]$.

To conclude the proof, we must first show that for $\lambda, \lambda' \in [0,1]$, d_λ does not dominate $d_{\lambda'}$; and second, that a mechanism given by $\mu d_\lambda + (1-\mu) d_{\lambda'}$ is undominated for $\mu \in [0,1]$.

For a given λ the utility of the low value individual is given by $\underline{U}(\lambda) = \alpha(\lambda)(v^L - \nu^L(\lambda))$, so that,

$$\underline{U}(\lambda) = (v^H - \frac{1}{N}) \times \frac{\lambda((N-1)v^H + v^L - 1)}{(v^H - v^L + \lambda((N-1)v^H + v^L - 1))},$$

which is increasing in λ, since the expression $\frac{\lambda((N-1)v^H+v^L-1)}{(v^H-v^L+\lambda((N-1)v^H+v^L-1))}$ is increasing in λ.

The utility of a high value individual is given by $\bar{U}(\lambda) = \alpha(\lambda)(v^H - \nu^H(\lambda))$, that is,

$$\bar{U}(\lambda) = \frac{(v^H - \frac{1}{N})}{(N-1)} \times \frac{\lambda((N-1)v^H + v^L - 1)}{(v^H - v^L + \lambda((N-1)v^H + v^L - 1))},$$

which is decreasing in λ, since $\frac{(1-\lambda)((N-1)v^H+v^L-1)}{(v^H-v^L+\lambda((N-1)v^H+v^L-1))}$ is decreasing in λ. Therefore, for $\lambda \neq \lambda'$, d_λ and $d_{\lambda'}$ do not dominate each other, one way or another.

Finally, it is easily shown that for $\lambda, \lambda' \in [0,1]$ and $\mu \in (0,1)$,

$$\mu\bar{U}(\lambda) + (1-\mu)\bar{U}(\lambda') > \bar{U}(\mu\lambda + (1-\mu)\lambda'),$$

while

$$\mu\underline{U}(\lambda) + (1-\mu)\underline{U}(\lambda') > \underline{U}(\mu\lambda + (1-\mu)\lambda').$$

Therefore, if the mechanism is given by $\mu d_\lambda + (1-\mu)d_{\lambda'}$, that is, d_λ with probability μ and $d_{\lambda'}$ with probability $1-\mu$, then such a randomized mechanism is also robust efficient. □

As a corollary, note that there is no classically efficient decision rule in D.

Corollary 2. *In the present environment, $D^* \cap \mathcal{D}^* = \emptyset$.*

Proof. This follows from the previous Proposition 2 and Theorem 8. □

While this first example serves mainly for illustrative purposes, the next example is less intuitable from known results.[20]

[20] An extension of the present example, which might have a particular appeal, is to consider a more elaborate setting where individual rationality must be satisfied only by some majority of individuals. While potentially impinging on some individuals' rights to freely opt out of the scheme, such a possibility might enhance efficiency of the resulting robust efficient decision rules.

4 Example 2: Property rights

In the second example, which relates to several classical problems, there are N individuals who must allocate an object among themselves. There are no *a priori* property rights so that $Y_0 = N \cup \{0\}$, where $y_0 = i$ when the object is allocated to some individual i and $y_0 = 0$ when the object is not allocated. Each individual's value of the object is privately known and is given by $\theta_i \in \Theta$ for some finite set Θ. None of the allocations incur any costs so that I omit the cost function c from the specification of the environment. I again assume a quasilinear setting, where $u_i(y, \theta) = y_i + 1_{\{y_0 = i\}}\theta_i$. I also assume that no individual can be forced to participate in the scheme and that there is no outside subsidizing agency. The feasible set D is thus given by the decision rules satisfying incentive compatibility, individual rationality, and budget balance.

When it is common knowledge that the individual who values the object most knows that, this problem is known as the "King Solomon's Dilemma." Then classically efficient allocations exist, even when no transfers between the individuals are allowed, see e.g., Perry and Reny (1999), Olszewski (2003) and Bag and Sabourian (2005). Another related problem is the so-called partnership dissolution problem where the individuals initially hold property rights over shares of the object. Under the *interim* (Bayesian) constraints Cramton et al. (1987) provide conditions whereby a classically efficient solution exists. See also McAfee (1991) for a related setting, McAfee (1992a) for an example without quasi-linearity, and especially the survey by Moldovanu (2002).

When the object is allocated, transfers are made solely between the individuals. For classical efficiency these transfers must sum to zero – otherwise some numeraire would be disposed of. When reports are θ denote by $\varphi_i(\theta)$ the probability that the object is allocated to i by $y_i(\theta)$ the transfer to/from i. Proposition 3 states that in general, there does not exist a classically efficient solution to this problem.

Proposition 3. *Let $N = 3$. Suppose that $\Theta_i = \Theta$, $\forall i$, $\{0, v^L, v^M, v^H\} \subset \Theta$. If $v^L < \frac{2}{3}v^M$, then $\nexists d \in D$, such that d is Pareto efficient.*

Proof. Suppose to the contrary that there exists a classically efficient mechanism. By Theorem 8 of Section 6 it suffices to consider anonymous mechanisms. Note that in a classically efficient allocation, budget balance implies exact budget balance.

The proof now utilizes Table 1 below which describes probabilities φ_i and transfers y_i for different classes of type profiles θ. In the first row, there are 10 different classes of profiles, up to permutations of individuals, and each class is denoted by a superscript. In the second row, if in a given class θ_i can take several different values, then these values are described by the appropriate set, e.g., in the class θ^1, individual 3's value $\theta_3 \in \{v^M, v^L, 0\}$. A draw of types θ in the class θ^k is denoted by $\theta \in \theta^k$. Finally, α is the probability that the object is allocated to the individual with the second highest allocation when her value equals v^M and there are no ties; β denotes that same probability when her value equals v^L and there are not ties.

In a classically efficient mechanism the object is allocated to an individual with the highest value of the good with probability 1 so that $\alpha = \beta = 0$. The following 5 steps demonstrate that the feasibility constraints pin down the allocation to that specified in Table 1. Consequently, no classically efficient mechanism exists in D.

Step 1. By symmetry and Pareto efficiency, at $\theta^0, \theta^4, \theta^{10}$, $\varphi_i = \frac{1}{3}$, and $y_i = 0, \forall i$. Additionally, set $\varphi_i(0,0,0) = 0, \forall i$, which is admissible as it is then efficient to dispose of the object. Note that setting $\varphi_i(0,0,0) = 0$ will yield highest efficiency as it provides the strongest possible incentives for higher types to not misrepresent downwards.

Step 2. At θ^1, $\varphi_1 = \varphi_2 = \frac{1}{2}$, $\varphi_3 = 0$, $y_1 = y_2 = -\frac{1}{6}v^M$, and $y_3 = \frac{1}{3}v^M$.

Proof. Consider individual 3. When $\theta_1 = \theta_2 = v^H$, as long as $\theta_3 < v^H$, since $\varphi_3 = 0$, individual

3 must receive the same transfer regardless of her report as she would otherwise have incentives to misrepresent her value for those values of θ_3 where she obtains the smallest compensation. Therefore, the allocation is indeed constant at θ^1.

Next, individual 3 can be compensated at most $\frac{v^H}{3}$ when $\theta_3 = v^M$ in order for her not to misrepresent her value to v^M when $\theta = \theta^0$. On the other hand, at $\theta \in \theta^1$, for 3 not to misrepresent her value to v^H, she must be compensated at least $\frac{v^M}{3}$. Since the transfers must sum to 0, and by symmetry, we have $y_i(\theta^1) = -\frac{y_1(\theta^1)}{2}, i < 3$. We thus obtain,

$$\varphi_3(\theta^3) = 0, y_3(\theta^1) \in [\frac{v^M}{3}, \frac{v^H}{3}], \tag{10}$$

$$\varphi_i(\theta^1) = \frac{1}{2}, y_i(\theta^1) = -\frac{y_1(\theta^1)}{2}, i < 3. \tag{11}$$

Consider θ^2. In order for individual 2 not to misrepresent her value to v^H, i.e., to θ^1, it must be that,

$$\frac{1}{2}v^M + y_2(\theta^1) \leq y_2(\theta^2) = y_3(\theta^2), \tag{12}$$

where the last equality follows by symmetry. Therefore, by strict budget balance

$$y_1(\theta^2) = -2(\theta^2) \geq -\frac{2}{3}v^M, \tag{13}$$

where the last inequality follows from (10), (11), and (12), since the most negative transfer from 1 at θ^2 is attained when $y_3(\theta^1) = \frac{v^M}{3}$ in which case $y_1(\theta^2) = -\frac{2}{3}v^M$.

Consider θ^4. If at θ^4 individual 1 misrepresents her value to θ^2, then she obtains $v^M + y_1(\theta^2) \geq \frac{1}{3}v^M = u_1(\theta^4)$. Hence, it must be that $y_1(\theta^2) = \frac{2}{3}v^M$.

Step 3. The allocation at θ^6 follows by a similar argument to that in Step 2, substituting θ^4 for θ^0, θ^5 for θ^1, and θ^{10} for θ^4.

Step 4. Recall that $\alpha = \beta = 0$. At θ^3, the allocation follows by considering a deviation of individual 2 to θ^1. At θ^7 the allocation follows by considering a deviation of individual 2 to θ^3; alternatively, since $\alpha = \beta = 0$, $\theta_1 = v^H$, and $\theta_3 = 0$, it must be that individual 2 obtains the same transfer regardless of whether she reports $\theta_2 = v^L$ or $\theta_2 = v^M$.

Step 5. At θ^9, the allocation follows by considering a deviation of individual 2 to either θ^7 or θ^3. Now observe that $y_2 = y_3 = \frac{1}{3}v^M$, hence $y_1 = -\frac{2}{3}v^M$, which is independent of whether individual 1 reports v^H, v^M, or v^L as in any of these cases she obtains the object so that her transfer must be unaffected. Therefore, since $v^H < \frac{2}{3}v^M$, the utility of 1 at $\theta_1 = v^L$ is negative, that is, $u_1 = v^H - \frac{2}{3}v^M < 0$, which contradicts the individual rationality.

By steps 1-5 there does not exist an anonymous mechanism satisfying the desiderata. The proof follows from Theorem 8. □

	θ_1	θ_2	θ_3	φ_1	φ_2	φ_3	t_1	t_2	t_3
θ^0	v^H	v^H	v^H	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^1	v^H	v^H	$\{v^M, v^L, 0\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^M}{6}$	$-\frac{v^M}{6}$	$\frac{v^M}{3}$
θ^2	v^H	v^M	v^M	1	0	0	$-\frac{2v^M}{3}$	$\frac{v^M}{3}$	$\frac{v^M}{3}$
θ^3	v^H	v^M	$\{0, v^L\}$	$1-\alpha$	α	0	$v^M\left(\alpha - \frac{2}{3}\right)$	$\frac{v^M}{3} - \alpha v^M$	$\frac{v^M}{3}$
θ^4	v^M	v^M	v^M	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^5	v^M	v^M	$\{v^L, 0\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^L}{6}$	$-\frac{v^L}{6}$	$\frac{v^L}{3}$
θ^6	$\{v^M, v^H\}$	v^L	v^L	1	0	0	$-\frac{2v^L}{3}$	$\frac{v^L}{3}$	$\frac{v^L}{3}$
θ^7	$\{v^M, v^H\}$	v^L	0	$1-\beta$	β	0	$-t_2 - t_3$	$(\alpha-\beta)\,v^L + \frac{v^M}{3} - \alpha v^M$	$\frac{v^L}{3}$
θ^8	v^L	v^L	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{v^L}{6}$	$-\frac{v^L}{6}$	$\frac{v^L}{3}$
θ^9	$\{v^L, v^H, v^M\}$	0	0	1	0	0	$-\frac{2v^M}{3} + 2\alpha v^M$	$\frac{v^M}{3} - \alpha v^M$	$\frac{v^M}{3} - \alpha v^M$
θ^{10}	v^L	v^L	v^L	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
θ^{11}	0	0	0	0	0	0	0	0	0

Table 1.

Along similar arguments, but with substantially more algebra and notation, it can be shown that in this environment there does not exist a classically efficient mechanism in the general case when $N \geq 2$. Note that finiteness of the type space is not the root cause of the problem in this example: the problem is that when the difference between the lowest non-zero value and 0 is too small, the individuals who do not obtain the object cannot be sufficiently compensated by the sole highest-value winner of the object, since not enough value is generated from the allocation to simultaneously satisfy incentive compatibility, budget balance, and individual rationality. This problem only gets worse when types can take a continuum of values.

In contrast to the *interim* setting of Cramton et al. (1987), Proposition 3 implies that a

Pareto efficient way to dissolve a partnership in a robust setting does not exist. The reason is that if the individuals initially hold property rights, then it is even more difficult to satisfy the individual rationality constraints. This is true under symmetric and asymmetric scenarios alike, that is, whether the initial shares are equal or not. A separate special case is when the object is initially owned by one of the individuals, similar to a resale of an object that had previously been acquired by one of the individuals in an auction.[21] One might intuit that a feasible classically efficient decision rule akin to a second-price auction ought to exist. Proposition 3 implies this is not the case.

I now construct an example of an anonymous robust efficient mechanism in this environment, see Definition 11 of Section 6 for the definition of an anonymous decision rule.[22] There are many ways in which Pareto efficiency may be relaxed while satisfying feasibility. An obvious possibility is, in some instances, to assign the good with a positive probability to the individual with the second-highest value; Such an assignment is still more efficient than assigning the good to some other individual with a lower value. Assigning the good in this way directly reduces the payments that must be made by the highest-value individual when she wins the object, and indirectly reduces the payments by differentiating between the incentive constraints of the second-highest and lower-value individuals.

Proposition 4. *Let $N = 3$. Suppose that $\Theta_i = \{0, v^L, v^M, v^H\}$, $\forall i$, and $v^L < \frac{2}{3}v^M$. Then there is an anonymous robust efficient mechanism which assigns the object to the second highest-value individual with probability $\alpha = \frac{1}{2v^M}(\frac{2}{3}v^M - v^L)$ when there are no ties. That mechanism is described in Table 1, setting $\beta = \alpha$.*

[21] Zheng (2002) provides an example of a classically efficient auction with resale under the *interim* constraints. In the present case, however, the environment is slightly different, in that the original owner does not value the object at zero, as would an auctioneer in a standard auction environment. It is indeed an implication of the results in Cramton et al. (1987) that even under the *interim* constraints, there does not exist a classically efficient decision rule in the present case.

[22] Note that as in the public good example of Section 3, while incentive compatible, a VCG mechanism will not simultaneously satisfy *ex-post* individual rationality and budget balance. Thus, VCG mechanisms are out of question here too.

Proof. Consider again Table 1. The proof follows along the lines of the proof of Proposition 3 with no differences in Steps 1-3, and the following distinctions:

In Step 4, at θ^3, when $\varphi_2 = \alpha > 0$, then that lowers the necessary transfer to individual 2 (still determined from the deviation of 2 to θ^1); this in turn determines the transfer to individual 2 at θ^7, as noted in Table 1.

In Step 5, at θ^9 in order for the individual 2 to not deviate either to θ^3 or θ^7, it must be that,

$$y_2(\theta^9) \geq \max\{(\alpha - \beta)v^L + (\frac{1}{3} - \alpha)v^M, (\frac{1}{3} - \alpha)v^M\}$$

By setting $(\frac{1}{3} - \alpha)v^M = v^L$ and $\beta \geq \alpha$, that ensures that at θ^9, the individual rationality constraint of 1 holds with equality, while minimizing α subject to the constraint that 2 would not want to misrepresent from θ^9 to either θ^3 or θ^7; by setting $\beta = \alpha$ the loss in the aggregate surplus is minimized also at θ^7 so that the mechanism cannot be Pareto dominated by any mechanism which satisfies incentive compatibility along with all the other constraints.

Finally, to assure incentive compatibility, we must set $\varphi_i = 0$ at $\theta = (0, 0, 0)$, which has no effect on Pareto efficiency. □

Apart from the aforementioned classical problems, the present example also evidently relates to an auction. Namely, in a setting where the seller in an auction is strategic in the sense of possessing private information regarding how much she values the object, the present example demonstrates that there is in general no classically efficient robust auction mechanism for allocating the object and that there exist robust efficient trading mechanisms (i.e., on the Pareto efficient frontier of feasible decision rules) that in some instances with some probability allocate the object to the second-highest bidder. For the present purpose, the above example illustrates the concept of constrained efficiency under robustness in an environment with private values and a private good (private consumption).

5 Robustness, *ex ante*, *interim* constrained efficiency

In their formulation of robust mechanism design, Bergemann and Morris (2005) provide a framework for mechanism design in the context of rich type spaces.[23] In this section, I use that framework to define *ex ante* and *interim* notions of constrained efficiency for such robust considerations. I then show that in any commonly studied environment these notions coincide with robust efficiency defined in Section 2.

As in Bergemann and Morris (2005), a type space is a collection,

$$\mathcal{T} = (T_i, \hat{\theta}_i, \hat{\pi}_i)_{i=1}^n, \text{ where,}$$

$t_i \in T_i$ is individual i's type, $\hat{\theta}_i : T_i \to \Theta_i$, so that $\hat{\theta}_i$ is i's payoff type when his type is t_i, and $\hat{\pi}_i : T_i \to \bar{T}_{-i}$, so that $\hat{\pi}_i(t_i)$ is the hierarchy of i's beliefs when his type is t_i. To sum up, the economy is now completely specified by the list,

$$\Gamma = (N, Y, \mathcal{T}, u),$$

and Γ is assumed to be common knowledge. In addition, in each state t, each individual i knows her private information t_i. Given a type space $\mathcal{T}$ we again define a decision rule d as a mapping $d : T \to \bar{Y}$. Note that the formulation of Section 2 can be embedded in this more general framework: let $\mathcal{T}$ be the payoff type space, i.e., $\mathcal{T}_i \equiv \Theta_i$. On the other hand, a decision rule d on $\boldsymbol{\Theta}$ naturally induces a decision rule d_T on T. As before, the set of all decision rules is given by $\mathcal{D}$.

[23] See, e.g., Mertens and Zamir (1985) and Brandenburger and Dekel (1993) for a general definition and discussion of rich type spaces.

In this setting, we can imagine different *ex ante* and *interim* notions of (constrained) efficiency. One can then vary the type space and ask what decision rules are constrained efficient in a sense that is robust to such variation. To define the domination relations we fix the type space $\mathcal{T}$.[24]

Given a type space $\mathcal{T}$ and a decision rule d, the individual i's corresponding *ex ante* and *interim* expected utilities are given by,

$$U_i^{\mathcal{T}}(d) = \int_{t\in\mathcal{T}} u_i(d(t),\hat{\theta}(t))\mathbf{d}\hat{\pi}_i(t),$$

$$U_i^{\mathcal{T}}(d \mid t) = \int_{t_{-i}\in T_{-i}} u_i(d(t),\hat{\theta}(t))\mathbf{d}\hat{\pi}_i(t_{-i} \mid t_i).$$

Definition 5. *A decision rule d' ex ante dominates d on $\mathcal{T}$, denoted $d' \blacktriangleright^{\mathcal{T}} d$, if,*

$$U_i^{\mathcal{T}}(d') \geq U_i^{\mathcal{T}}(d), \quad \forall i \in N, \tag{14}$$

with at least one strict inequality; d' interim dominates d on $\mathcal{T}$, denoted $d' \rhd^{\mathcal{T}} d$, if,

$$U_i^{\mathcal{T}}(d' \mid t) \geq U_i^{\mathcal{T}}(d \mid t), \quad \forall t_i \in T_i \forall i \in N, \tag{15}$$

with at least one strict inequality.

I remark that the notion of *ex post* domination on $\mathcal{T}$ is defined as in Section 2 and denoted by $\succ^{\mathcal{T}}$.

For our purposes, it will be enough to limit attention to two sorts of type spaces: *all full*

[24]In the following definitions it is implicitly assumed that the type space $\mathcal{T}$ is finite, that is, that each T_i is finite. This will be enough for our purposes here. The definitions can easily be extended to more general type spaces, e.g., the universal type space, as long as T is a Hausdorff space, which is the case as long as $\mathbf{\Theta}$ is compact, see Mertens and Zamir (1985).

support common prior payoff type spaces, defined as in Bergemann and Morris (2005); and *all full support subjective priors payoff type spaces*, which I introduce here.[25] A type space $\mathcal{T}$ is a payoff type space if $T_i = \Theta_i$ and $\hat{\theta}_i$ is the identity map, $\forall i$. A payoff type space satisfies a *full support common prior* assumption if in addition $\exists p \in int(\bar{\mathbf{\Theta}})$, such that,

$$\hat{\pi}_i(t_i)[t_{-i}] = p(t_{-i} \mid t_i), \;\; \forall t_i \in T_i, \forall i \in N.$$

A payoff type space satisfies a *full support subjective priors*[26], if, $\exists (p_1, \ldots, p_n) \in int(\bar{\mathbf{\Theta}})^n$, such that,

$$\hat{\pi}_i(t_i)[t_{-i}] = p_i(t_{-i} \mid t_i), \;\; \forall t_i \in T_i, \forall i \in N.$$

The assumption of full support is maintained throughout, so that when we refer to, e.g., a subjective prior payoff type space, what is really meant is a full support subjective prior payoff type space and no confusion should arise. I also remark that the set of all subjective prior payoff type spaces is, of course, a superset of the set of all common prior payoff type spaces. From now on, the type spaces under consideration will be either of these two payoff type spaces.

A decision rule d on a payoff type space $\mathcal{T}$ is naturally a decision rule on $\mathbf{\Theta}$. Moreover, if a given (constrained) set of decision rules $D \subset \mathcal{D}$ is feasible for all subjective priors payoff type spaces, then it is sensible to think of D as a set of feasible decision rules on $\mathbf{\Theta}$. This will be the case if all the constraints specifying the decision rules in D are satisfied (or not) on all the subjective priors payoff type spaces; for example, all the *ex post* constraints specified in Section 2 evidently have this property. For such decision rules and feasible sets D all the definitions

[25] One could also consider the universal type space, where T_i is the set of all i's *coherent* hierarchies of beliefs, see also Mertens and Zamir (1985) or Brandenburger and Dekel (1993). Indeed, our definition of robust efficiency is applicable to the *universal type space*, and our main result of this section, Theorem 3 below, holds on the universal type space *a fortiori*.

[26] To formally represent a subjective priors type space in the belief hierarchy, where each individual has a subjective prior over the payoff types, and these subjective priors are common knowledge, such a type space would be infinite and different from the payoff type space. Nevertheless, such a subjective priors type space is homeomorphic to the representation given here.

of Section 2 apply.[27] When considering all subjective priors payoff type spaces, heuristically, each individual does not know the others' subjective priors. Fix a $D \subset \mathcal{D}$ and assume that it is feasible for all subjective priors payoff type spaces.

Definition 6. *A decision rule d' uniformly ex ante dominates d, if $d' \blacktriangleright^{\mathcal{T}} d$, on all common prior payoff type spaces $\mathcal{T}$; denote $d' \blacktriangleright^{CP} d$. A $d \in D$ is uniform ex ante constrained efficient in D if $\nexists d' \in D$, s.t., $d' \blacktriangleright^{CP} d$.*

A decision rule d' uniformly interim dominates d, if $d' \triangleright^{\mathcal{T}} d$, on all common prior payoff type spaces $\mathcal{T}$; denote $d' \triangleright^{CP} d$. A $d \in D$ is uniform interim constrained efficient in D if $\nexists d' \in D$, s.t., $d' \triangleright^{CP} d$.

Denote by $D^{\blacktriangleright}$ and $D^{\triangleright}$ the sets of uniform *ex ante* and *interim* constrained efficient decision rules in D, respectively.

Definition 7. *A decision rule d' robust ex ante dominates d, if $d' \blacktriangleright^{\mathcal{T}} d$, on all subjective priors payoff type spaces $\mathcal{T}$; we denote $d' \blacktriangleright^{SP} d$. A $d \in D$ is weak ex ante constrained efficient in D if $\nexists d' \in D$, s.t., $d' \blacktriangleright^{SP} d$.*

A decision rule d' robust interim dominates d, if $d' \triangleright^{\mathcal{T}} d$, on all subjective priors payoff type spaces $\mathcal{T}$; we denote $d' \triangleright^{SP} d$. A $d \in D$ is weak interim constrained efficient in D if $\nexists d' \in D$, s.t., $d' \triangleright^{SP} d$.

Denote by $\tilde{D}^{\blacktriangleright}$ and $\tilde{D}^{\triangleright}$ the sets of weak *ex ante* and *interim* constrained efficient decision rules in D, respectively.

The more difficult it is for a decision rule to dominate another decision rule, the larger

[27] For a general type space $\mathcal{T}$ it is not true that every decision rule d on T is also a decision rule on $\boldsymbol{\Theta}$ – two type profiles $t, t' \in T$ may pertain to the same profile of payoff types, and d may assign different allocations to t and t'. As for the feasibility of decision rules, one could also take various sorts of constraints that do depend on the specific type space, derive the corresponding type-space specific feasible sets of decision rules, and then consider the intersection of these sets over the relevant type spaces. See Bergemann and Morris (2005) and also Ledyard (1978) for an extended discussion of this issue, especially regarding the incentive compatibility constraints.

the set of undominated decision rules. Therefore, the following relationships hold:

$$D^{\blacktriangleright} \subset \tilde{D}^{\blacktriangleright} \subset \tilde{D}^{\triangleright} \subset D^{*} \text{ and } D^{\blacktriangleright} \subset D^{\triangleright} \subset \tilde{D}^{\triangleright} \subset D^{*}. \tag{16}$$

Theorem 3. *A decision rule d' ex post Pareto dominates d, if and only if, d' uniformly ex ante dominates d.*

Proof. It is evident that if d' *ex post* Pareto dominates d, then d' uniformly *ex ante* dominates d. For the converse, suppose that d' does not *ex post* Pareto dominate d. For each $i \in N$, let $\mathbf{\Theta} = A_{i,d'} \cup A_{i,d} \cup A_{i,d'd}$, where $u_i(d'(\theta),\theta) > u_i(d(\theta),\theta)$, $\forall \theta \in A_{i,d'}$, $u_i(d(\theta),\theta) < u_i(d'(\theta),\theta)$, $\forall \theta \in A_{i,d}$, and $u_i(d(\theta),\theta) = u_i(d'(\theta),\theta)$, $\forall \theta \in A_{i,d'd}$. Note that there exists at least one i such that $A_{i,d} \neq \emptyset$. Now take $p \in int(\bar{\mathbf{\Theta}})$, such that p stacks most of the probability mass on $A_{i,d}$. Therefore,

$$\int_{\theta \in \mathbf{\Theta}} u_i(d'(\theta),\theta)\mathbf{d}p(\theta) < \int_{\theta \in \mathbf{\Theta}} u_i(d(\theta),\theta)\mathbf{d}p(\theta),$$

so that d' does not uniformly *ex ante* dominate d. □

Since the strongest and the weakest domination relations coincide, we have the following corollary.

Corollary 4. *For a $D \subset \mathcal{D}$, all the sets in (16) are identical,*

$$D^{\blacktriangleright} = \tilde{D}^{\blacktriangleright} = D^{\triangleright} = \tilde{D}^{\triangleright} = D^{*}.$$

As far as *interim* implementability of decision rules is concerned, Bergemann and Morris (2005) suggest that the requirement that a decision rule be implementable on all full support common prior payoff type spaces is a minimal requirement for robustness. If this criterion is applied to the question of efficiency, the minimal requirement for robustness is that a decision rule d be uniformly *interim* constrained efficient, that is, $d \in \tilde{D}^{\triangleright}$. By Theorem 3 and Corollary 4 that is equivalent to requiring that d be robust efficient in D.

6 Feasibility, common knowledge, durability, and anonymity

In this section, I touch upon several additional considerations. I have chosen to place these considerations at the end, as I view these considerations supplementary, or in some cases also too broad to be discussed here in greater depth.

6.1 Incentive compatibility and feasibility

Among the feasibility constraints, incentive compatibility stands out – it is necessary to impose incentive compatibility or some other strategic constraint as a feasibility constraints on decision rules. Why is that so? The answer lies in the revelation principle, more precisely its contrapositive: if a decision rule is not incentive compatible, then there does not exist *any* institution (or game form) such that the decision rule is the result of an equilibrium behavior of the individuals interacting through that institution. I illustrate this on the example of *ex post* incentive compatibility.[28]

Consider an environment given by $(N, Y, \boldsymbol{\Theta}, c, u)$ and a game form $\Gamma \equiv (N, \boldsymbol{\Theta}, O, S, \gamma, \tilde{u})$. Here S is the set of players' type-contingent strategies (possibly mixed), so that an $s \in S$ is a mapping $s : \Theta \to O$, where $s_i(\theta) = s_i(\theta_i)$; O is the outcome space of game form Γ, γ is the outcome mapping, $\gamma : O \to \bar{Y}$, and $\tilde{u} : S \times \Theta \to \mathbb{R}^n$ are the players' indirect utility functions,

[28] By no means do I wish to suggest that *ex post* incentive compatibility constraints are the only strategic feasibility constraints consistent with robustness. For example, Yamashita (2015) considers a notion of rationalizability which is independent of the individuals' prior beliefs, and Börgers and Smith (2012) consider a similar procedure. However, my main concern here is with efficiency and I will limit the discussion of strategic constraints to the example of *ex post* incentive compatibility.

$\tilde{u}(s,\theta) = u(\gamma(s),\theta), s \in S$. In Γ, a strategy profile s is an *ex post* Nash equilibrium if,

$$\tilde{u}_i(s,\theta) \geq \tilde{u}_i(s'_i, s_{-i}, \theta), \forall s'_i \in S_i, \forall \theta \in \Theta.$$

Note that a decision rule is a particular kind of game form where $O = \Theta$ and $S_i = \{s_i : \Theta_i \rightarrow \bar{\Theta}_i\}$. The revelation principle can now be stated as follows.[29]

Theorem 5. *If there exists an ex post Nash equilibrium s of a game form $\Gamma = (N, \boldsymbol{\Theta}, O, S, \gamma, \tilde{u})$, then there exists an ex post incentive compatible direct revelation mechanism d, such that $d(\theta) = \gamma(s(\theta))$, for all $\theta \in \Theta$.*

Proof. Define a decision rule d by $d(\theta) = \gamma(s(\theta))$ and suppose that d is not *ex post* incentive compatible. That is, $\exists \tilde{\theta} \in \Theta$, $i \in N$, and $\theta'_i \in \Theta_i$, such that,

$$u_i(d(\tilde{\theta}), \tilde{\theta}) < u_i(d(\theta'_i, \tilde{\theta}_{-i}), \tilde{\theta}).$$

Now define i's strategy $s'_i \in S_i$ in Γ by $s'_i(\theta_i) = s_i(\theta_i), \forall \theta_i \neq \tilde{\theta}_i$ and $s'_i(\tilde{\theta}_i) = s_i(\theta'_i)$. Therefore,

$$\tilde{u}_i(s'_i, \tilde{\theta}) = u_i(d(\theta'_i, \tilde{\theta}_{-i}), \tilde{\theta}) > u_i(d(\tilde{\theta}), \tilde{\theta}) = \tilde{u}_i(s_i, \tilde{\theta}),$$

which is a contradiction, since s was an *ex post* Nash equilibrium to begin with. □

Corollary 6. *Let EQ be a refinement of the ex post Nash equilibrium. If a decision rule d is not ex post incentive compatible, then there does not exist any game form Γ and a strategy profile s in Γ, such that s satisfies EQ, and $\gamma(s(\theta)) = d(\theta), \forall \theta \in \Theta$.*

Proof. Suppose d is incentive compatible and there exists a game form Γ and an s such that s

[29] The revelation principle is well known, see e.g., Myerson (1991), and I only state it here for clarity. Note also that I consider only partial implementation, as that is more relevant to present considerations. See, e.g., the survey by Jackson (2003) and the chapter by Palfrey (2002) on the difference between partial versus full implementation.

satisfies EQ and $d(\theta) = \gamma(s(\theta)), \forall\theta \in \Theta$. Since EQ is a refinement of *ex post* Nash equilibrium, if s satisfies EQ then, in particular, s is an *ex post* Nash equilibrium strategy profile. By Theorem 5 the decision rule d is incentive compatible, a contradiction. □

The message of Corollary 6 is that a decision rule which is not incentive compatible cannot be thought of as a result of the individuals' equilibrium (in the sense of *ex post* equilibrium) behavior in any institution, be it static or dynamic. In a dynamic setting, as long as the equilibrium concept is some refinement of the *ex post* Nash equilibrium, such statement holds *a fortiori*. Similarly, this is true for any other equilibrium notion for which the revelation principle holds, e.g., dominant strategy equilibrium and the corresponding notion of incentive compatibility, *strategy-proofness*, and in a non-robust framework, Bayes-Nash equilibrium and the corresponding *interim* incentive compatibility.[30]

6.2 R

obust common knowledge and durability

Whether and how the individuals may change from, or settle on, various decision rules is a quintessential question in the discussion of efficiency. Holmström and Myerson (1983) call this aspect of a decision rule durability. As described by Holmström and Myerson (1983) in the context of *interim* incentive efficient decision rules, the issue is two-fold: on the one hand, there may exist an informational state such that the allocation prescribed by a given decision rule (or, *a decision*) is Pareto dominated by some other allocation; on the other hand, unanimous agreement to change to a different decision rule would constitute a common knowledge event, whereby all individuals prefer the latter decision rule. Holmström and Myerson (1983) show

[30]Such refinements of Bayes-Nash equilibrium for dynamic settings are the sequential equilibrium and the subgame-perfect Bayes-Nash equilibrium. See Myerson (1986) for an early application of these notions to mechanism design. A recent general framework for applying these notions to mechanism design may be found in Pavan et al. (2014).

that *interim* incentive efficiency is equivalent to non-existence of a common knowledge event whereupon the individuals unanimously prefer another feasible decision rule over a given feasible decision rule. The notion of robust efficiency is not immediately amenable to such considerations – common knowledge, in the sense of Aumann (1976) is defined in terms of the individuals' subjective prior beliefs. Hence, it is not obvious how to define a common knowledge event in the language of Section 2. The definition of robust *interim* constrained efficient decision rules provides a natural conduit to address these issues.

Given a subjective priors payoff type space $\mathcal{T}$, as in Holmström and Myerson (1983) an event $R \subset T \equiv \mathbf{\Theta}$ is common knowledge on $\mathcal{T}$ if $R = R_1 \times \ldots \times R_n$, $R_i \subset T_i$, and,

$$p_i(t'_{-i} \mid t_i) = 0, \quad \forall t \in R, \quad \forall t' \notin R, \ \forall i. \tag{17}$$

Definition 8. *An event R is robust common knowledge if it is common knowledge on every subjective priors payoff type space.*

Intuitively, if R is robust common knowledge, then, as long as the individuals' (rich) types are in R, all the individuals assign zero probability to the types outside R regardless of what subjective prior over the payoff type space each individual holds.

Definition 9. *A decision rule d' robust interim dominates d within R, denoted $d' \rhd_R d$, if, $R \neq \emptyset$ and,*

$$U_i^{\mathcal{T}}(d' \mid t) \geq U_i^{\mathcal{T}}(d \mid t), \quad \forall t \in R, \ \forall i, \tag{18}$$

with at least one strict inequality, for all subjective priors payoff type spaces $\mathcal{T}$.

Suppose now that each individual may have a variety of different subjective priors, that is, consider all subjective priors payoff type spaces. If at some point prior to the realization of their private information – their types – the individuals agree on the decision rule that they should use, then they should settle on some *weakly ex ante* constrained efficient decision rule. By Theorem 3, such decision rule would be robust efficient. Equivalently, this would be the case if the decision rule were at such prior stage proposed by a benevolent social planner without knowledge of the individuals' priors; by Corollary 4, this would be the case even if the individuals had some common prior unknown to the social planner. However, under the former decentralized interpretation, it seems more sensible to assume that the individuals' priors were subjective.

It seems equally sensible to assume that the individuals should only consider rules that satisfy some notion of incentive compatibility. Bergemann and Morris (2005) showed that if a decision rule were *interim* incentive compatible on every common prior payoff type space, then such decision rule would be *ex post* incentive compatible as defined in Section 2. Therefore, if the decision rule were *interim* incentive compatible on every subjective priors payoff type space, then it would be *ex post* incentive compatible, *a fortiori*. In a decentralized discussion the individuals should then consider decision rules satisfying *ex post* incentive compatibility as well as any other constraints dictated by the economic environment at hand.

The question then is whether the individuals could agree to change from a given robust efficient incentive compatible decision rule after their private information has been realized. The following theorem is analogous to the characterization of *interim* incentive efficient decision rules in Holmström and Myerson (1983).

Theorem 7. *An incentive compatible decision rule d is robust efficient if and only if there does not exist a robust common knowledge event R and an incentive compatible decision rule d', such that $d' \rhd_R d$.*

Proof. For sufficiency, let $R = T$ and the result follows by Corollary 4. For necessity, suppose that $d' \rhd_R d$, where d' is incentive compatible and R is a robust common knowledge event.

Define

$$d^*(t) = \begin{cases} d'(t), & \text{if } t \in R, \\ \\ d(t), & \text{if } t \notin R. \end{cases}$$

Evidently $d^* \rhd^{SP} d$, and moreover, d^* is incentive compatible. To see this, take an i and first consider $t_i \in R_i$. He would not want to misrepresent to any type in R_i since d' is incentive compatible. Now fix a prior p_i. Since d' is *ex post* incentive compatible, it is *interim* incentive compatible under the prior p_i, and since $d' \rhd_R d$, d' *interim* dominates d under the prior p_i so that i's average payoff under p_i is at least as high under d' than under d. Since d is also *interim* incentive compatible under p_i, it follows that given a prior p_i, i would not have any incentives to misrepresent outside R_i. Now consider the case when $t_i \notin R_i$. Then, since R is robust common knowledge, it follows that the $t_{-i} \notin R_{-i}$, so that the decision rule coincides with d, which is *ex post* incentive compatible. Therefore, for every prior p_i, d^* is *interim* incentive compatible so that it is *ex post* incentive compatible. This implies that d is not robust efficient, a contradiction. □

Theorem 7 suggests that if the individuals are engaged in a robust efficient decision rule d , then they could not unanimously agree to change to another incentive compatible decision rule. That is true at least to the extent that it could not be common knowledge on all subjective priors type space that the individuals unanimously prefer a different decision rule. One could still posit that if each individual learned some additional information beyond his type, then all individuals might be willing to change to some other decision rule. On the one hand, under *ex post* incentive compatibility the individuals would have no incentives to misrepresent their types, even if they knew *all* the private information in the economy. Furthermore, robust efficient decision rules are precisely those that cannot be improved upon in every informational state *and still satisfy incentive compatibility.* Therefore, a heuristic intuition is that at least under some reasonable specifications of decentralized deliberations, the answer to the above question is no. Lacking a formal specification of such a decentralized discussion, I leave this as an intuition.

6.3 S

ymmetry and anonymity

Another related possibility is to consider additional characteristics of the economic environment such as symmetry. A Rawlsian argument suggests that in a symmetric environment, *ex ante* equal individuals may be prone to elect at least a symmetric lottery over robust efficient decision rules; under conditions of risk aversion, or under a *max-min* social welfare criterion, the individuals should presumably prefer a deterministic choice of an *anonymous* decision rule over such a lottery. Anonymity and symmetry are well-known desiderata, especially when it comes to concerns of fairness, see e.g., Moulin (1993) and Serizawa (1999). Symmetry requires that two individuals with the same preferences and initial endowments receive the same allocation. Anonymity requires that when names of individuals are permuted, so are their consumption bundles. To keep the language consistent with the literature I refer to anonymous decision rules on the one hand, and to symmetric environments and symmetric sets of decision rules on the other. The symmetry axioms concern both, the environment and the set of decision rules, rather than a specific decision rule. In particular, non-anonymous decision rules may, in a symmetric environment, belong to a symmetric set of decision rules.

While the main concern here has been with the environments where classically efficient rules do not exist, that is, where $D^* \cap \mathcal{D}^* = \emptyset$, in some environments, even imposing efficiency might still lead to a substantial indeterminacy regarding transfers, or *prices*. In a symmetric environment one way to resolve this indeterminacy is to consider symmetric, or more general anonymous decision rules.

Suppose that for each i, $\Theta_i = \Theta$, so that $\boldsymbol{\Theta} = \Theta^n$. Denote by $\Pi(n)$ the set of all permutations of n elements. A $\pi \in \Pi(n)$ may denote either a permutation of the individuals in N, or the corresponding permutation of the elements of vector $\theta \in \Theta^n$, or the corresponding permutation of the individuals' transfers $y_N \in \mathbb{R}^n$; furthermore, $\pi(\theta)$ denotes the permuted vector θ and $\pi(i)$ denotes the image of an element $i \in N$ under permutation π. For a decision rule d and a permutation π denote by d^π the decision rule $d \circ \pi$, that is, $d^\pi(\theta) = d(\pi(\theta))$, $\theta \in \boldsymbol{\Theta}$.

Definition 10. *An environment is symmetric if $\Theta_i = \Theta, \forall i$, and for every $y \in Y$, and every permutation $\pi \in \Pi(n)$, there exists a $y_0' \in Y_0$ such that $u_{\pi(i)}(y_0', \pi(y_N), \pi(\theta)) = u_i(y, \theta)$, $\forall i, \forall \theta \in \boldsymbol{\Theta}$.*

Both examples of sections 3 and 4 correspond to symmetric environments.

Definition 11. *Let the environment be symmetric. A decision rule $d \in \mathcal{D}$ is anonymous if, for every permutation π of N,*

$$u_i(d(\theta), \theta) = u_{\pi(i)}(d(\pi(\theta)), \pi(\theta)), \forall i, \forall \theta.$$

Definition 12. *The set of decision rules D is symmetric if the environment is symmetric and $d \in D \Rightarrow d^\pi \in D$, $\forall d \in D$, for all permutations π.*

To illustrate the aforementioned indeterminacy of prices, consider a symmetric environment. There, existence of classically efficient decision rules implies that there also exist anonymous classically efficient decision rules. While the assumption of symmetry is somewhat restrictive, it covers several classical allocation problems. In particular, it applies to the two examples in sections 3 and 4.

Theorem 8. *Suppose the environment is symmetric, quasi-linear, and $D \subset \mathcal{D}$ is convex. Then there exists a classically efficient decision rule in D, if and only if, there exists an anonymous classically efficient decision rule in D.*

Proof. For sufficiency, an anonymous classically efficient decision rule is classically efficient. For necessity, take a $d \in D^* \cap \mathcal{D}^*$ and suppose d is not symmetric. For each $\pi \in \Pi(n)$, d^π is also classically efficient and $d^\pi \in D$. To show this, suppose to the contrary that there existed a d' such that d' Pareto dominated d^π; in that case $d'^{\pi^{-1}} = d' \circ \pi^{-1}$ would Pareto dominate d, a contradiction. Therefore, $d^\pi \in D^* \cap \mathcal{D}^*, \forall \pi \in \Pi(n)$. Now let $d^*(\theta) = \frac{1}{|N|!} \sum_{\pi \in \Pi(n)} d^\pi(\theta)$. By convexity $d^* \in D$. It is also evident that d^* is symmetric. Finally, recall our assumption that the decision rules in $\mathcal{D}$ cannot entail infinite subsidies, that is, $\sum_{i \in N} d_i(\theta) \leq K$, $\forall \theta \in \boldsymbol{\Theta}$, for all $d \in D$ and some constant K. Along with quasi-linearity, this implies that every classically efficient decision rule in $\mathcal{D}$ must yield the same total sum of the individuals' utilities, and any feasible decision rule yielding such a total sum of utilities is classically efficient. Hence, $d^* \in D^* \cap \mathcal{D}^*$. □

One could further ask under what conditions utilitarianism leads to an anonymous decision rule. That may be of interest and perhaps relevant to the current discussions in public economics, inequality, and the distribution of wealth.

7 Appendix

For a given set of decision rules $D \subset \mathcal{D}$ denote by $\mathcal{U}[D] = \{U^d(.) \mid d \in D\} \subset \mathbb{R}^{|\boldsymbol{\Theta}| \times n}$, i.e., $\mathcal{U}[D]$ is the set of utility allocations for all types of all agents, arising from the decision rules in D. Clearly, D satisfies convexity if and only if $\mathcal{U}[D]$ is convex. Similarly, D is closed if and only if $\mathcal{U}[D]$ is closed in $R^{|\boldsymbol{\Theta}| \times n}$. Finally, note that since transfers are bounded by assumption, and all the other sets are finite, $\mathcal{U}[D]$ is bounded in $R^{|\boldsymbol{\Theta}| \times n}$, for any D.

We first show that for any D satisfying convexity, the set $\mathcal{U}[D^*]$ satisfies a weaker property. For $a, a' \in \mathcal{U}[D^*]$, define $S(a, a') = \{\mu \in (0,1) \mid \mu a + (1-\mu)a' \in \mathcal{U}[D^*]\}$.

Lemma 1. *For each $a, a' \in \mathcal{U}[D^*]$, either $S(a, a') = (0,1)$ or $S(a, a') = \emptyset$.*

Proof. Take $a, a' \in \mathcal{U}[D^*]$ and let $d, d' \in D^*$ be the corresponding scf's. Define $d_\alpha \equiv \alpha d + (1 - \alpha)d'$, for $\alpha \in [0, 1]$. Let $\bar{S} = (0, 1) \setminus S(a, a')$, i.e., $\bar{S}$ is the set of α's such that $\alpha d + (1 - \alpha)d' \notin D^*$.

Assume that $\bar{\alpha} \in \bar{S}$, for some $\bar{\alpha}$, hence $\bar{S} \neq \emptyset$. Let $d' \succ d_{\bar{\alpha}}$. Now take convex combinations of d' and d to dominate all d_α, s.t. $\alpha \leq \bar{\alpha}$, and take convex combinations of d' and d' to dominate all d_α, s.t. $\alpha \geq \bar{\alpha}$. Thus, $\bar{S} \neq \emptyset \Rightarrow \bar{S} = (0, 1)$. □

Proof of Theorem 1. To see that $D^{\mathcal{W}} \subset D^*$ assume that $\exists d \in D^{\mathcal{W}}$ which solves (2) for some $\lambda(.)$ and $\exists d' \in D^{\mathcal{W}}$, s.t., $d' \succ d$. By the definition of $\succ$, inserting d' into the optimization program (2), we see that its value is higher than that obtained from d, for all $\lambda(.)$ and all $Pr \in int(\bar{\mathbf{\Theta}})$, a contradiction.

For the converse, we proceed in 5 steps.

Step 1. $\mathcal{U}[D^{\mathcal{W}}]$ has empty interior in $R^{|\mathbf{\Theta}| \times n}$.

Assume the opposite and take an open ball $o(a, \epsilon) = \{a' \mid ||a - a'||_2 < \epsilon\} \subset \mathcal{U}[D^{\mathcal{W}}]$, for some $\epsilon > 0$. Let d be the scf corresponding to the point a. Taking $a + \frac{\epsilon}{2}(1, 1..., 1)$ and letting d' be the corresponding scf we obtain $d' \succ d$, a contradiction.

Step 2. $\mathcal{U}[D^{\mathcal{W}}]$ is closed. This follows immediately from D closed.

Step 3. Either $\mathcal{U}[D^{\mathcal{W}}]$ has empty interior relative to $\mathcal{U}[D]$, or $\mathcal{U}[D^{\mathcal{W}}] = \mathcal{U}[D]$.

This follows from convexity of $\mathcal{U}[D]$ and Lemma 1.

Step 4. Take a direction $\alpha \in R^{|\mathbf{\Theta}| \times n}$, $||\alpha||_2 = 1$, and let the correspondence $a(\alpha)$ be defined as the solution to the linear program,

$$a(\alpha) = \arg\max_{a \in \mathcal{U}[D]} \alpha.a,$$

where $\alpha.a$ is the standard scalar product between the two vectors. Then,

$$\mathcal{U}[D^{\mathcal{W}}] = cl(\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}| \times n}} a(\alpha)).$$

By compactness of $\mathcal{U}[D^{\mathcal{W}}]$, $a(\alpha) \neq \emptyset, \forall \alpha$. By convexity of $\mathcal{U}[D]$, it is clear that if $int_{R^{|\mathbf{\Theta}|\times n}}(\mathcal{U}[D]) \neq \emptyset$, then $bo(\mathcal{U}[D]) = \cup_{\alpha \in R^{|\mathbf{\Theta}|\times n}} a(\alpha)$, where $bo(.)$ denotes the boundary of the set, i.e., the set of all its limit points which are not in its interior; and if $int_{R^{|\mathbf{\Theta}|\times n}}(\mathcal{U}[D]) = \emptyset$, then $\mathcal{U}[D]) = \cup_{\alpha \in R^{|\mathbf{\Theta}|\times n}} a(\alpha)$. If $int_{\mathcal{U}[D]}(\mathcal{U}[D^{\mathcal{W}}]) = \emptyset$, then since $\mathcal{U}[D^{\mathcal{W}}]$ is closed (Step 2), $\mathcal{U}[D^{\mathcal{W}}] = cl(\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}} a(\alpha))$.[31]

On the other hand, if $\mathcal{U}[D^{\mathcal{W}}] = \mathcal{U}[D]$, then by Step 1 and convexity of $\mathcal{U}[D], \mathcal{U}[D^{\mathcal{W}}]$ is a compact and convex linear subset of $R^{|\mathbf{\Theta}|\times n}$. Hence there exists a $\bar{\alpha} \in R_{++}^{|\mathbf{\Theta}|\times n}$, s.t. $\mathcal{U}[D^{\mathcal{W}}] = a(\bar{\alpha})$, (this $\bar{\alpha}$ must be strictly positive by the definition of $\succ$), so that $\mathcal{U}[D^{\mathcal{W}}] = a(\bar{\alpha}) \subset \cup_{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}} a(\alpha) \subset \cup_{\alpha \in R^{|\mathbf{\Theta}|\times n}} a(\alpha) = \mathcal{U}[D] = \mathcal{U}[D^{\mathcal{W}}]$, and the claim follows.

Step 5. Fix weights $\lambda(\theta) = (\lambda_1(\theta), ..., \lambda_n(\theta))$. Consider the linear program,

$$\arg\max_{d \in D} \sum_{\theta \in \mathbf{\Theta}} \sum_{i \in N} \lambda_i(\theta) u_i(\theta) =$$

$$\arg\max_{a \in \mathcal{U}[D]} \sum_{\theta \in \mathbf{\Theta}} \sum_{i \in N} \lambda_i(\theta) a_{\theta,i}.$$

Now observe that $\{(\lambda_i(\theta))_{\theta \in \mathbf{\Theta}, i \in N} \mid \lambda \in R_{++}^{|\mathbf{\Theta}|\times n}\} = \{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}\}$, which proves the theorem. $\square$

We remark that fixing $\bar{Pr} \in int(\bar{\mathbf{\Theta}})$, then

$$\{\big(\lambda_{\theta,i} \bar{Pr}(\theta)\big)_{\theta \in \mathbf{\Theta}, i \in N} \mid \lambda \in R_{++}^{|\mathbf{\Theta}|\times n}\} = int(\{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}\}).$$

[31] Observe that even in this case, one may construct examples such that $\mathcal{U}[D^{\mathcal{W}}] = \cup_{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}} a(\alpha)$, i.e., where $\cup_{\alpha \in R_{++}^{|\mathbf{\Theta}|\times n}} a(\alpha)$ is closed. That is not the case whenever $bo(\mathcal{U}[D])$ is a smooth manifold.

On the other hand,

$$\{(\lambda_i Pr(\theta))_{\theta\in\mathbf{\Theta},i\in N} \mid \lambda \in R^n_{++}, Pr \in int(\bar{\mathbf{\Theta}})\} \neq \{\alpha \in R^{|\mathbf{\Theta}|\times n}_{++}\}.$$

Observe also that if either the assumption of full support, $Pr \in int(\bar{\mathbf{\Theta}})$, or the assumption that all the welfare weights must be strictly positive is dropped, then one can construct examples where $D^{\mathcal{W}}$ is a strict subset of $\mathcal{D}^*$.

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