The role of the elasticity of substitution in an endogenous growth model of structural change

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Abstract: *We propose a multi-sector model of endogenous growth in which structural change occurs according to a price effect between intermediates inputs. The unique final good, which can be consumed or used as capital, is produced using two intermediate goods, into a CES production function. The two intermediate sectors differ in terms of knowledge accumulation, in sector 1 workers devote a fraction u of their time endowment to knowledge accumulation "à la Romer"; in sector 2 they produce only. In this framework we can consider intermediate goods as complement or substitute; this differentiation leads to two different paths for structural change. We characterize two different Non-Balanced Growth Paths (NBGP) with respect to the value of the elasticity of substitution between the two intermediate goods which are consistent with structural change. Labor and capital are indeed reallocated in the knowledge intensive sector when inputs are substitutes, and in the other sector when they are complements. The aggregate behavior of this model, along the equilibrium path, is also consistent with the Kaldor facts. Additionally we show the persistence of inequalities, in human and physical capital, through the "manifold of steady-states" feature during the growth process, and that there is conditional convergence.*

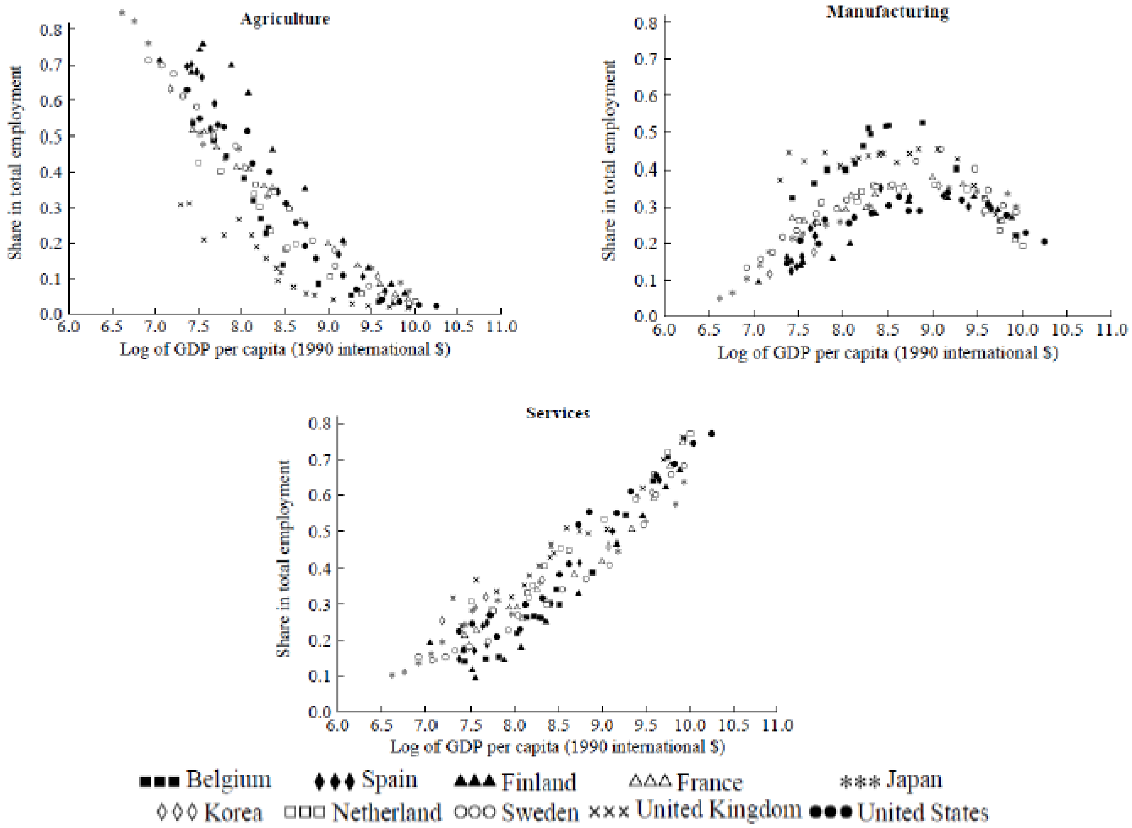
1 Introduction

Since 1957 and the work of Kuznets there were a huge amount of research dealing with structural change and trying to explain the mechanism behind this. Along the 20^{th} century we saw a drop of labor share in agriculture in favor of the service sector (manufacturing sector is characterized by an hump-shape) and economic theory tried to bring an explanation. While in the 60's economist were focused on unbalanced growth as Baumol [3] or Hansen [7], in the 2000's there were a new interest for structural change as a stylized fact and a new literature came along in macroeconomics with the article of Kongsamut, Rebelo and Xie [10]. With our work we explore a new way of explanation for structural change by building a multi-sector model of endogenous growth, based on endogenous human capital accumulation. We have 3 sectors, one final good sectors using the two others as intermediate inputs. We introduce accumulation of technical knowledge in the first intermediate sector only, TFP is absent from the second. The idea is that there are some sectors with a huge potential growth based on human capital in the field of big data, communication, green energies, ... and some more "classical" sectors in which it seems hard to improve productivity by training of workers as automobile industry, mining, retail trade, etc ... The interaction between the two intermediate sectors depends on the elasticity of substitution (if sectors are substitutes or complements) in the CES production function for the final good. As it is standard in the literature, our model is consistent the Kaldor facts: growth rates, capital-output ratio, share of capital income in GDP and the real interest rate are constant over time.

Features of structural change are well desccribed by the working paper of Herrendorf, Rogerson and Valentinyi [8], they documented very well the existing literature on structural change, differentiating between models based on income effects or price effects. We provide a brief summary of the existing literature in section 2. They also produce a complete empirical analysis of stylized facts associated to this literature, using EU-KLEMS data. We present below three of their graphs about reallocation of Labor across agriculture, manufacturing and services.

Our model is mainly inspired by the forthcoming paper of Ghiglino, Nishimura and Venditti (GNV) [6]; and by the paper of Acemoglu and Guerrieri (AG) [1]. We use the structure of GNV and we extend it by adding a CES production function for the unique final good, instead of a Cobb-Douglas one, and we use the methodology of AG to solve analytically the model. The main difference between GNV and AG is in the modelisation of technical progress, for the first paper progress is endogenous, and for the second it is exogenous. We think model of endogenous growth are more efficient to represent reality, and could give better explanation to structural change. Introducing elasticity of substitution in the GNV model creates interesting relationships between our two sectors, the results are completely different if inputs are substitutable or complementary. A further work to improve this paper should be an empirical investigation of the different values for the elasticity of substitution between sectors.

The model is build as follow, the final good, that can be consumed or used as capital, is produced using two intermediates output as input, into a CES production function. The two inputs are produced in two competitive sectors using Cobb-Douglas technologies, the first sector will be the only one to deal with knowledge intensity. We characterize this by a trade-off between working



source: Herrendorf, Rogerson and Valentinyi (2013)

Figure 1: Kuznets facts

and knowledge accumulation: in sector 1, agents devote a fraction u of their time endowment to production, the fraction $(1-u)$ is devoted to knowledge accumulation. We apply this knowledge, denotes A, to the labor of workers in sector 1, it represents productivity gain. For simplification, all the new workers in sector 1 get all the knowledge available at date t.

Contrary to the initial idea of Kuznets, we only consider 2 sectors, for the moment it's impossible to rely our model with empirical regalauities or data on agriculture, manufacturing and services. Our intermediates sectors differ in terms of knowledge intensities, we consider an extreme case in which knowledge accumulation occurs only in one sector. This separation of intermediate sectors don't allow us to deal with agriculture, manufacturing and services, but it's still possible to confront the model to data.

Results depend strongly on the value of ϵ, the elasticity of substitution. In the substitutable case, we show there exists a unique set of non-balanced growth path compatible with structural change and the Kaldor facts, locally stable, exhibiting saddle-point configuration. In the reverse case, it is still possible to obtain a saddle-point but we need to impose a condition on ϵ and α, the labor intensity in sector 1. Stability requires $\alpha > \epsilon$, if this condition does not hold, there is complete instability of the model. We will see this below but the idea behind is that, in a multi-sector model with capital deepening, there is a trade-off between investing in one sector instead of the other. In this paper, when inputs are complement, we need to allocate more intensively labor and capital in sector 2 to compensate the lack of endogenous growth, but if inputs are not enough complementary, it becomes more efficient to invest in sector 1 and we are not able to reach a stable growth path.

Adding an elasticity of substitution to GNV paper add a sectoral dominance, as in AG. If $\epsilon > 1$ sector 1 is the dominant, it means that asymptotically all the labor and the capital will be allocated in this sector. If $\alpha > \epsilon$ sector 2 will be the dominant. If $1 > \epsilon > \alpha$ we cannot conclude, there is instability of the model. The parameter ϵ tend to be the key variable in this analysis, instead of technical knowledge, this is why a good identification of this parameter is crucial if we what to apply this kind of model. A quick overview of US and French data tend to show that knowledge intensive sector is substitutable to non-knowledge intensive one. Using Occupational Employment Statistics (OES) Survey from 1999 to 2016 show the following dynamic for United states:

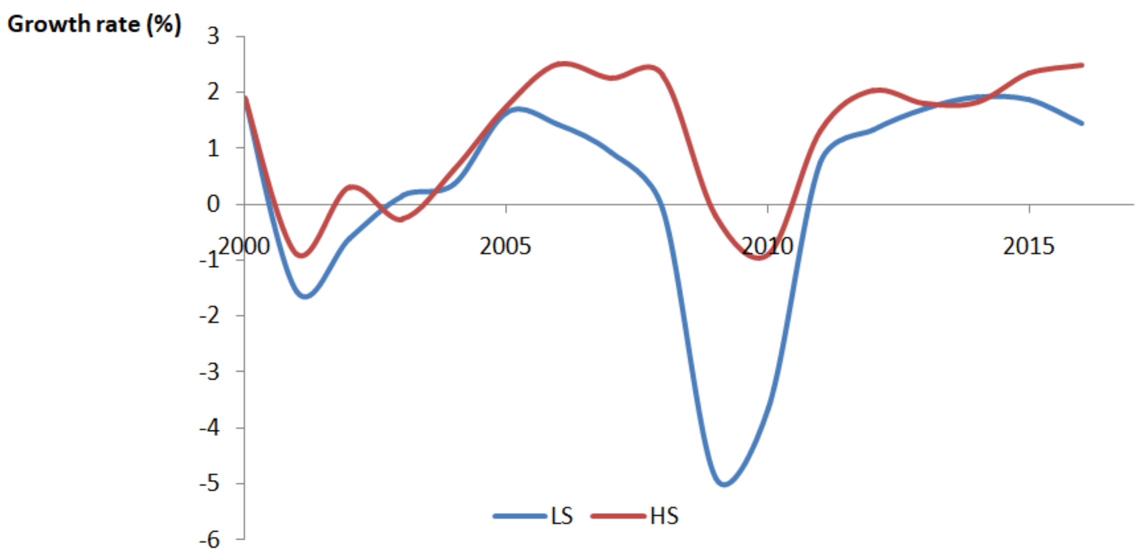


Figure 2: Low-skilled and High-skilled growth rates in United States

For this kind of purpose, time horizon is not enough but the classification in OES survey is useful for our purpose. We take job classification from Bureau of labor Statistics in the OES survey, from 11-0000 (Management occupation) to 29-0000 (Healthcare Practitioners and Technical Occupations) as high-skilled workers and the rest of the classification as low-skilled (we find Personal Care and

Service Occupations, Construction and Extraction Occupations, ...). The number of High-skilled workers in United states grows faster than low-skilled individuals, and it seems that High-skilled workers suffer less from crisis (2001 and 2008). According to our model, this is consistent with an elasticity of substitution larger than one between high-skilled sectors and low-skilled sectors.

For France we use INSEE data ”Recensement de la population 1968-2014”, for workers from 25 to 54 years old, these data consist in 7 surveys (1968, 1975, 1982, 1990, 1999, 2009, 2014) and individuals are classified according to their ”catégorie socio-professionelle”, we consider ”cadres” as high-skilled workers and the rest of the population as low-skilled. The dynamic in France from 1968 to 2014 is as follow:

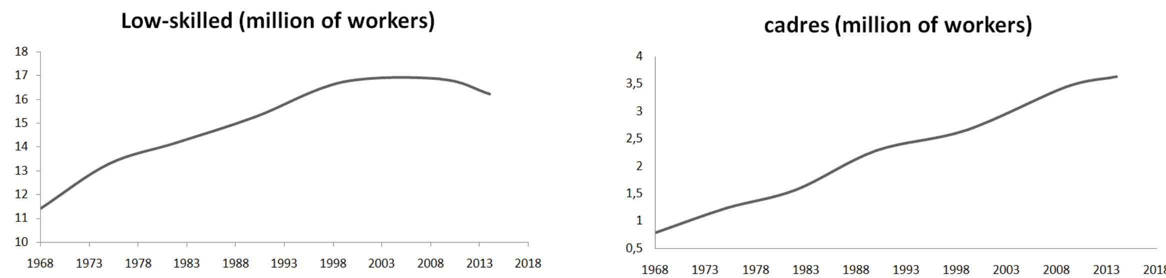


Figure 3: Number of low and high skilled workers in France

We cannot compute growth rate as for United states because it is a 7 point graphic, and we do not have the detailed variation year by year, but we have the global dynamic of skills for french workers from 1968 to 2014. The number of ”cadres” is multiply by 5 during the considered period when low-skilled workers are multiplied by approximatively 1.5. Although is seems low-skilled workers were touched by 2008 crisis, unlike high-skilled workers. French data are also in favor of an elasticity of substitution larger than 1.

These graphics serve to illustrate the dynamic in two developed countries for two different time horizons, they come from an arbitrary cut-off between workers. Nevertheless, it seems to fit our theory. A deeper analysis of data, and a more transparent way to discriminate between workers are needed to have a complete empirical counterpart.

Into our local stability analysis we also exhibits a classical feature of endogenous growth model: the manifold of steady-states. This comes from Luca’s seminal paper [12], in endogenous growth model there exist an infinity of steady-states and the economy reach a particular one according to its initial endowments. In our model, even if all economies reach the same Non Balanced Growth Path they will not be at the same level of development, it depends on knowledge and capital endowment, (k_0, a_0), of the country. If there are inequalities at the beginning, they remain through time even if all country share the same structure, there is conditional convergence of economies. Even for the manifold of steady states, elasticity of substitution play an important role. When inputs are substitutable there is a linear relationship between the steady state values of a ans k, when inputs are complement this relationship is parabolic. An higher endowment of technical knowledge leads to an higher level of capital if inputs are complements. It is in line with the rest of the paper and the necessity of capital accumulation to compensate the lack of endogenous growth in sector two.

The rest of the paper is organized as follows. Section 2 presents a brief overview of the existing literature related to the structural change. Section 3 introduces our model, its static equilibrium and the dynamic, ending with our non-balanced growth path. Section 4 deals with the local stability properties and the manifold of steady states. Finally, section 5 concludes and section 6 contains the appendix with all our mathematical proofs.

2 Literature review

Here is a brief summary of the literature about structural change:

- Kongsamut, Rebelo, Xie [10]. In this paper structural change is only driven by Income effect, this comes from the utility specification choose by the authors. They use a non homothetic utility function, with the following terms $\bar{A}$, the subsistence consumption level and $\bar{S}$, the home production of services. They obtain changes into consumption shares but constant relative prices. They proved the existence of a balanced growth path configuration, with sectoral labor reallocation. It's one of the first paper investing the idea of structural change, and its coexistence with balanced growth.

- Foellmi, Zweimuller [5]. This paper introduce also structural change through a pure income effect. Instead on focusing on 3 broad sectors, authors introduce a mass of potential consumption. They specified preferences such that margin utility is finite at zero consumption and decreases to 0 for a finite satiation level. Increase of income over time will increases the mass of consumption, there will be adjustments in both intensive and extensive margins. It's due to the "hierarchical" consumption specification of their model. The utility function is designed such that each good has a different priority level, as long as the revenue will grow, individuals will consume more goods, the extensive margin, and higher quantities of goods already consumed, intensive margins.

Goods i don't differs in their production function, only on the demand side. Structural change will occurs with the extensive effect on goods production. An higher number of goods coincide with a movement of the labor force in the new sectors of production. Authors are able to reproduce stylized facts of Agriculture, Manufacturing and services. If we considered low i's goods as basics needs, agricultural goods, medium i's as manufactured goods and higher i's as services; their model provide a good framework to study structural change.

- Ngai, Pissarides [14]. This paper deals with the strict opposite of Kongsamut, Rebelo, Xie. Here the authors try to relate structural change only on relative price effects. Their model is quite standard, they assume the same production function in all sectors, except for TFP growth, they use different technological accumulation rates in order to generate relative price changes. They've found that difference between sector's rates of price change is equal to the difference between their TFP growth rates.

- Boppart [4]. This paper deals with non-gorman preferences (PIGL class of preferences) in such way to study the effect of both relative price effect and income effect to explain the structural change. An important contribution of this paper is the will to focus on these 2 effects instead of a single effect. It's why model specification is very important in this paper. The central point of the analysis is the design of preferences. T. Boppart considers a continuum of heterogeneous agents (in terms of initial wealth), their utility is given by an additively separable representation of inter-temporal preferences, with a particular type of indirect utility function: Price Independent Generalized Linearity (PIGL), based on the work of Muellbauer (1975,1976). Technology side is quite standard with 2 consumption sectors, goods and services, and 1 investment good which can be transformed one-to-one in capital. Productions functions of goods and services differ only in their exogenous rate of technical progress.

The heart of its analysis is in the utility specification, this choice allows the author to have a representative agent, in Muellbauer's sense, and to derive aggregate demand. With this model's structure we have a decreasing share of goods across time and an increasing share of services, generated by both a price effect and an income effect.

3 The model

3.1 Technology, preferences and social planner problem

We consider an economy in which at each time t there is a continuum $[0, N(t)]$ of infinitely-lived agents which are characterized by homogeneous preferences. We assume a standard formulation for the utility function which is compatible with homothetic growth such that

$$\int_0^{+\infty} \frac{c_i(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt \tag{1}$$

where $c_i(t)$ is consumption of agents of type $i \in [0, N(t)]$ at time t, $\theta > 0$ is the inverse elasticity of intertemporal substitution in consumption and $\rho > 0$ is the discount factor. We assume that total population grows at the constant exponential rate $n \in [0, \rho)$, so that $N(t) = e^{nt}N(0)$. Agents are heterogeneous with respect to their allocation of time endowment, they choose in which sector they want to work.

There are two sectors in this economy, with three factors of production: capital, $K(t)$, labour, $L(t)$, and individual technological knowledge, $A(t)$. We do not consider any externality. Final output, $Y(t)$, is produced as an aggregate of the output of two intermediate sectors, $Y_1(t)$ a "knowledge-intensive sector" and $Y_2(t)$. The numbers of workers $N_1(t)$ and $N_2(t)$ in these two sectors together with their respective growth rates $\dot{N}_1(t)/N_1(t) = g_{N_1}(t)$ and $\dot{N}_2(t)/N_2(t) = g_{N_2}(t)$ are endogenously determined at the equilibrium.

Agents working in the "knowledge-intensive sector" devote a fraction $u(t) \in (0,1)$ of their unit of time to production. Total labour in this sector is $L_1(t) = u(t)N_1(t)$. The rest of time $1-u(t)$ is devoted to the accumulation of individual technological knowledge $A(t)$. Each agent entering the knowledge-intensive sector at any time t_0 acquires the available knowledge $A(t_0)$. We assume that

$$\dot{A}(t) = z[1-u(t)]A(t) - \eta A(t) \tag{2}$$

with $z > 0$ and $\eta > 0$ the depreciation rate of knowledge. On the other hand, agents working in the second sector will spend all their unit of time to work so that total labour in this sector is $L_2(t) = N_2(t)$.

The final good is produced through a CES technology such that

$$Y(t) = \left(\gamma Y_1(t)^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma)Y_2(t)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

with $\gamma \in (0,1)$ and ϵ the constant elasticity of substitution. With a CES production function, the choice of the value of elasticity of substitution is crucial, if we choose $\epsilon > 1$ inputs are substitutes, if $\epsilon < 1$ inputs are complements. According to which case we consider, results are strictly differents. We begin by considering inputs as substitutes, it seems to be a huge restriction but in the paper by Papageorgiou, Saam and Schulte [15] on the elasticity of substitution between clean and dirty inputs, they found an elasticity of substitution larger than one.

Sector 1 produces the "knowledge-intensive intermediate good" using capital, labour and knowledge through the following Cobb-Douglas technology:

$$Y_1(t) = [L_1(t)A(t)]^{\alpha} K_1(t)^{1-\alpha} \tag{4}$$

with $\alpha \in (0,1)$. The product $L_1(t)A(t) = u(t)N_1(t)A(t)$ then represents total efficient labour. Note that, considering capital and labour as inputs, the total factor productivity (TFP) is given by $A(t)^{\alpha}$. Sector 2 produces the second intermediate good using only capital and labour through the following Cobb-Douglas technology:

$$Y_2(t) = L_2(t)^{\beta} K_2(t)^{1-\beta} \tag{5}$$

with $L_2(t) = N_2(t)$ and $\beta \in (0,1)$. Contrary to the "knowledge-intensive" sector, the second sector has a constant TFP.

Denoting total capital by $K(t)$ and total labour by $L(t)$, capital and labour market clearing conditions require at each date $K(t) \geq K_1(t) + K_2(t)$ and $L(t) \geq L_1(t) + L_2(t) = N_1(t)u(t) + N_2(t)$. The capital accumulation equation is standard

$$\dot{K}(t) = Y(t) - \delta K(t) - N_1(t)c_1(t) - N_2(t)c_2(t) \tag{6}$$

where $\delta > 0$ is the depreciation rate of capital.

The central planner has a Benthamite objective function and considers the following intertemporal optimization problem

$$
\begin{array}{cl}
\max\limits_{\{c_i(t),K_i(t),L_i(t)\}_{i=1,2},u(t),A(t)} & \displaystyle\int_0^{+\infty} \left(N_1(t)\frac{c_1(t)^{1-\theta}}{1-\theta} + N_2(t)\frac{c_2(t)^{1-\theta}}{1-\theta} \right) e^{-\rho t} dt \\
s.t. & (2), (3), (4), (5), (6) \text{ and} \\
 & K(t) \geq K_1(t) + K_2(t) \\
 & L(t) \geq L_1(t) + L_2(t) = N_1(t)u(t) + N_2(t) \\
 & K(0), A(0), N(0) \text{ given}
\end{array}
\tag{7}
$$

The Hamiltonian in current value is (we omit subscript for t to simplify notations):

$$
\begin{aligned}
\mathbb{H} \;=\; & N_1 \frac{c_1^{1-\theta}}{1-\theta} + N_2 \frac{c_2^{1-\theta}}{1-\theta} + P_1 \Big[(L_1 A)^\alpha K_1^{1-\alpha} - Y_1 \Big] + P_2 \Big[L_2^\beta K_2^{1-\beta} - Y_2 \Big] \\
& + \; P \Big[(\gamma . Y_1(t)^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) . Y_2(t)^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}} - \delta K - N_1 c_1 - N_2 c_2 \Big] + Q \Big[z(1-u) - \eta \Big] A \\
& + \; \lambda \left[K - K_1 - K_2 \right] + \mu [L - L_1 - L_2]
\end{aligned}
$$

with $L_1 = uN_1$, where P is the price of aggregate capital, P_1 the price of the knowledge-intensive good, P_2 the price of the second good, Q is the price of knowledge and λ and μ are the Lagrange multipliers associated to the capital and labour market clearing conditions.

3.2 Static equilibrium

We begin with the static equilibrium of this model, the first order conditions with respect to the control variables $c_1, c_2, u, L_1, L_2, K_1, K_2, Y_1$ and Y_2 give:

$$\begin{aligned} c_i^{-\theta} &= P \text{ for any } i = 1, 2 && (8) \\ P_1 \alpha \frac{Y_1}{L_1 A} N_1 &= Qz && (9) \\ P_1 &= P\gamma \left(\frac{Y}{Y_1} \right)^{\frac{1}{\epsilon}} && (10) \\ P_2 &= P(1-\gamma) \left(\frac{Y}{Y_2} \right)^{\frac{1}{\epsilon}} && (11) \\ \lambda = P_1(1-\alpha)\frac{Y_1}{K_1} &= P_2(1-\beta)\frac{Y_2}{K_2} && (12) \\ \mu = P_1 \alpha \frac{Y_1}{L_1} &= P_2 \beta \frac{Y_2}{L_2} && (13) \end{aligned}$$

Substituting (10) and (11) into (12) and (13) gives

$$\begin{aligned} \gamma(1-\alpha)Y(t)^{\frac{1}{\epsilon}} K_1^{-1} Y_1^{\frac{\epsilon-1}{\epsilon}} &= (1-\gamma)(1-\beta)Y(t)^{\frac{1}{\epsilon}} K_2^{-1} Y_2^{\frac{\epsilon-1}{\epsilon}} && (14) \\ \gamma\alpha Y(t)^{\frac{1}{\epsilon}} L_1^{-1} Y_1^{\frac{\epsilon-1}{\epsilon}} &= (1-\gamma)\beta Y(t)^{\frac{1}{\epsilon}} L_2^{-1} Y_2^{\frac{\epsilon-1}{\epsilon}} && (15) \end{aligned}$$

Even if agents work in two different sectors, they all consume the same amount such that $c_1(t) = c_2(t) = P^{-\frac{1}{\theta}}$. It comes from the fact that in the "knowledge-intensive sector", the rental rate of capital and the wage rate are given by:

$$r_1 = (1-\alpha)\tfrac{Y_1}{K_1}, \quad w_1 = \alpha \tfrac{Y_1}{N_1 u A} \qquad (16)$$

while in the "second sector", r_2 and w_2 are given by

$$r_2 = (1-\beta)\tfrac{Y_2}{K_2}, \quad w_2 = \beta \tfrac{Y_2}{N_2} \qquad (17)$$

It's straightforward to see that:

$$w_1(t)A(t)P_1(t) = w_2(t)P_2(t)$$

$$r_1(t)P_1(t) = r_2(t)P_2(t) \qquad (18)$$

Is equivalent to equations (14) and (15). These two equalities imply that all agents have the same purchasing power. At the equilibrium we will be able to have individuals with the same inter-temporal profile of consumption.

Our social planner problem can be split to write the output maximization instead of the maximization of consumption. It allows us to obtain a set of optimal allocations into the static and dynamic equilibrium. Let us define the maximized value of output as a capital stock function:

$$\Phi(K(t),t) \quad = \quad \max_{K_1(t),K_2(t),L_1(t),L_2(t)} F[Y_1(t),Y_2(t)]$$

Subject to (4), (5) and to the market clearing conditions. And given $K(t) > 0$, $L(t) > 0$ and $A(t) > 0$.

In order to obtain the desired functional form, we introduce the following ratios:

$$\kappa(t) \equiv \frac{K_1(t)}{K(t)}, \qquad \lambda(t) \equiv \frac{L_1(t)}{L(t)} \tag{19}$$

We also have $1 - \kappa(t) \equiv \frac{K_2(t)}{K(t)}$ and $1 - \lambda(t) \equiv \frac{L_2(t)}{L(t)}$. And combining this statement with the equations (14) and (15) we obtain:

$$\kappa(t) = \left[1 + \left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_1(t)}{Y_2(t)}\right)^{\frac{1-\epsilon}{\epsilon}}\right]^{-1} \tag{20}$$

and

$$\lambda(t) = \left[1 + \left(\frac{1-\alpha}{1-\beta}\right)\left(\frac{\beta}{\alpha}\right)\left(\frac{1-\kappa(t)}{\kappa(t)}\right)\right]^{-1} \tag{21}$$

From equation (3) we see that we can write the maximized value $\Phi(K(t),t)$ such that:

$$Y(t) = \Phi(K(t),t) = \psi(t)A^{\alpha}\lambda(t)^{\alpha}\kappa(t)^{1-\alpha}L(t)^{\alpha}K(t)^{1-\alpha} \tag{22}$$

with $\quad \psi(t) \;=\; \gamma^{\frac{\epsilon}{\epsilon-1}}\left(1 + (\frac{1-\alpha}{1-\beta})\frac{1-\kappa(t)}{\kappa(t)}\right)^{\frac{\epsilon}{\epsilon-1}}$

It follows then:

$$R(t) = \Phi_K(K(t),t) = (1-\alpha)\psi(t)A^{\alpha}\lambda(t)^{\alpha}\kappa(t)^{-\alpha}L(t)^{\alpha}K(t)^{-\alpha}$$

3.3 Inter-temporal Equilibrium

Using the Hamiltonian in current value we obtain these two results:

$$\begin{aligned}\dot{P} &= \rho P - \frac{\partial \mathbb{H}}{\partial K} = P(\rho+\delta) - P(1-\alpha)\gamma Y_1^{\frac{\epsilon-1}{\epsilon}} Y^{\frac{1}{\epsilon}} K_1^{-1} \\ \dot{Q} &= \rho Q - \frac{\partial \mathbb{H}}{\partial A} = \rho Q - Qzu - Q(z(1-u)-\eta)\end{aligned}$$

Adding the law of motion of capital and knowledge accumulation characterize the full dynamic system:

$$\frac{\dot{P}}{P} = (\rho+\delta) - (1-\alpha)\gamma Y_1^{\frac{\epsilon-1}{\epsilon}} Y^{\frac{1}{\epsilon}} K_1^{-1} \tag{23}$$

$$\frac{\dot{Q}}{Q} = -(z-\rho-\eta) \tag{24}$$

$$\frac{\dot{A}}{A} = z(1-u)-\eta \tag{25}$$

$$\frac{\dot{K}}{K} = \frac{Y}{K} - \delta - \frac{N.P^{\frac{-1}{\theta}}}{K} \tag{26}$$

It follows that any path $\{K(t), A(t), P(t), Q(t)\}_{t\geq 0}$ that satisfies the conditions (20)-(26) together with the transversality conditions

$$\lim_{t\to+\infty} P(t)K(t)e^{-\rho t} = 0 \text{ and } \lim_{t\to+\infty} Q(t)A(t)e^{-\rho t} = 0$$

for any given initial conditions ($K(0)$, $A(0)$) is an optimal solution of problem (7).

The next step of our analysis is to provide a set of growth rates consistent with both Kaldor facts and Kuznets facts. This means the characterization of the Non Balanced Growth Path (NBGP). In our model the existence of an endogenous technical progress will bring the economy to an asymmetric growth path, it's the origin of the structural change, through the non-balanced growth rates. Kaldor facts ensure that all growth rates are constant over time.

In contrast to some other papers (like Boppart (2014), Kongsamut, Rebelo, Xie (2001), ...) the structural change comes from the production sector, instead of demand sector. It's a pure price effect from technology differences, as in Ngai and Pissarides [14].

Theorem 1. *When $\epsilon > 1$, there exists a unique set of non-balanced growth rates such that:*

$$\begin{aligned}
g_K &= g_Y = g_{K_1} = g_{Y_1} = n + \frac{1}{\theta}\Big(n - \rho + z - \eta\Big) \\
g_{Y_2} &= n + \frac{1-\beta\epsilon}{\theta}\Big(n - \rho + z - \eta\Big) \\
g_{K_2} &= n + \frac{(1+\beta-\beta\epsilon)}{\theta}\Big(n - \rho + z - \eta\Big) \\
n_1 &= n \\
n_2 &= n + \frac{(1-\epsilon)}{\theta}\beta\Big(n - \rho + z - \eta\Big) \\
g_P &= -(n - \rho + z - \eta) \\
g_Q &= \rho - z + \eta \\
g_A &= \frac{n - \rho + z - \eta}{\theta}
\end{aligned}$$

Additionally, the amount of time devoted to production is sector 1, u, is constant over time along the NBGP. Its equilibrium value is:

$$u^* = \frac{1}{z}\Big[z - \eta - \frac{1}{\theta}(n - \rho + z - \eta)\Big]$$

Proof: see appendix 6.1

We can extract a lot of implications from theorem 1, and first of all the great importance of the elasticity of substitution's value, ϵ. As long as we assume that intermediates inputs are substitutes the following property holds: $g_{K_1} > g_{K_2}$, $n_1 > n_2$ and $g_{Y_1} > g_{Y_2}$. From this we derive some interesting conclusions.

As in Acemoglu and Guerrieri we have a dominant sector in our economy: capital, labor and intermediate input grow faster in sector 1: the "knowledge-intensive sector" is the driver of growth. The level of knowledge, endogenously determined by the model, is the key element to produce growth, even if we have capital deepening, since $g_K = g_Y = g_{K_1} = g_{Y_1} = n + g_A$. Asymptotically, aggregate variables and sector 1 variables exhibit the same growth rates. Structural change occurs, $g_{K_1} > g_{K_2} \quad ; \quad n_1 > n_2$ labor and capital are reallocated in sector 1.

Intuitively, since we have technical progress in sector 1 we will produce more efficiently in sector 1. The gap between the productivity will foster production in the "knowledge intensive sector", which is the dominant in this economy. Along the development process we will substitutes relatively input 2 by input 1. In this model capital deepening is asymmetrically captured by the more efficient sector, $g_{K_1} > g_{K_2}$.

Another precision is about the existence of sector 2, according to the range of parameters we consider, it's possible to observe the end of sector 2. If $n < \frac{(1+\beta-\beta\epsilon)}{\theta}(n - \rho + z - \eta)$ (which could be compatible with an high elasticity of substitution) $g_{K_2} < 0$ and $n_2 < 0$: sector 2 will vanish asymptotically.

Even if $n > \frac{(1+\beta-\beta\epsilon)}{\theta}(n-\rho+z-\eta)$, steady states values of κ and λ are such that $\kappa^* = 1$ and $\lambda^* = 1$ (proof in appendix 6.1), which literally means that $K_1 = K$ and $L_1 = L$ when t tend to infinity. But these are just asymptotic properties, when $t \to \infty$ the sector 2 is relatively too small to be significant, but it has not vanished, its lower growth rate is the cause of its quasi naught relative weight.

The last implication is a decreasing price of the final good, $g_P < 0$. As we have endogenous technological progress we produce more efficiently and the price of the final good drops. If we decompose this price effect using equations (10) and (11), we have a decrease of price mainly due to the decreasing price of intermediate input 1, this price effect drives the structural change, reallocating capital and labor in sector 1.

If the two intermediates goods are complement instead of substitutes, i.e $\epsilon < 1$, we have $\lambda^* = 0$ and $\kappa^* = 0$. Growth rates of capital and labor are larger in the second sector than in the first one, our asymptotic results should be the reverse. We should have dominance of sector 2, but preliminary computations show that, when $\epsilon < 1$, theorem 1 becomes:

Theorem 2. *When $\epsilon < 1$, there exists a unique set of non-balanced growth rates such that:*

$$\begin{aligned}
g_K &= g_Y = g_{K_2} = n + g_A \\
n_2 &= n \\
g_{Y_2} &= n + (1-\beta)g_A \\
g_{Y_1} &= n + (1-\beta+\beta\epsilon)g_A > g_{Y_2} \\
g_{K_1} &= g_K - (1-\epsilon)\beta g_A < g_{K_2} \\
n_1 &= n - (1-\epsilon)\beta g_A < n_2 \\
g_A &= \frac{n-\rho+z-\eta}{\theta + (1-\epsilon)\beta\frac{\epsilon-1}{\epsilon}}
\end{aligned}$$

Additionally, the amount of time devoted to production is sector 1, u, is constant over time along the NBGP. Its equilibrium value is:

$$u^* = \frac{1}{z}\left[z - \eta - \frac{n-\rho+z-\eta}{\theta + (1-\epsilon)\beta\frac{\epsilon-1}{\epsilon}}\right]$$

We use the same methodology as for the previous theorem to prove this.

When $\epsilon < 1$, Labor and capital grow faster in sector 2, there will be relative reallocation of means of production in this sector. However, it is not the dominant in terms of input production: $g_{Y_1} > g_{Y_2}$. Even if sector 2 relatively attract labor and capital, "knowledge intensive" sector grows faster. Knowledge accumulation is more "growth productive" than capital deepening, additionally g_A is larger when $\epsilon < 1$, agents devote less time for production in sector 1 than in the substitutable case.

This is explained by the complementarity between intermediates goods, we need both of them to produce efficiently. As long as we have technical progress, which is constant along the NBGP, we need to compensate the gap in productivity by a faster accumulation of capital in sector 2. It's not the same case than in Acemoglu and Guerrieri [1], here we have endogenous growth because of knowledge accumulation, but we are in the extreme case where there is only one sector which deals with it. In our model growth is not lead by capital deepening but by the endogenous knowledge accumulation (g_A), even if capital and labor are mainly redirected to sector 2 we have $g_{Y_1} > g_{Y_2}$. Sector 1 is still the dominant.

The elasticity of substitution governs the direction of structural change in this model, capital and labor will be allocated differently according to their substitutability (complementarity). We will see in the next section the local stability properties of our 2 cases, and we will show that for the complementarity case stability of the system requires some restrictions on parameters.

4 Local stability properties

4.1 Substitutable intermediate inputs

In this section we will try to prove that our NBGP is the only converging path and it's locally stable. For our purpose we will reformulate our dynamical system given by equations (23)-(26) in its stationarized form by removing the constant growth rates from the variables. We obtain the following stationarized variables: $k(t) = K(t)e^{-g_K t}$, $a(t) = A(t)e^{-g_A t}$, $q(t) = Q(t)e^{-g_Q t}$ and $p(t) = P(t)e^{-g_P t}$, for all $t \geq 0$. From (24) we see that Q has a constant growth rate, this leads to $\dot{q}(t) = 0$. Our model will depends on the initials values q_0 and N_0.
We substitute these new variables into our dynamical system to obtain an equivalent stationarized system.

Proposition 1. *Along a stationarized equilibrium the dynamical system with respect to price p, capital k and knowledge a is the following:*

$$\frac{\dot{p}}{p} = -\left[(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}a^{\alpha}\left(\frac{L_1(k,a,p,q_0,N_0)}{K_1(k,a,p,q_0,N_0)}\right)^{\alpha} + g_P - \rho - \delta\right] \quad (27)$$

$$\frac{\dot{k}}{k} = \psi a^{\alpha}\left(\frac{L_1(k,a,p,q_0,N_0)}{K_1(k,a,p,q_0,N_0)}\right)^{\alpha}\kappa - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k} \quad (28)$$

$$\frac{\dot{a}}{a} = z\Big(1 - u(k,a,p,q_0,N_0)\Big) - g_A - \eta \quad (29)$$

Proof: see Appendix 6.2

For the last part we need to prove the existence of a unique steady state, for a given level $q_0 > 0$, $(k(q_0)^*, a(q_0)^*, p(q_0)^*)$ which will correspond to the NBGR in our theorem 1. Then we linearize our dynamical system (27)-(29) around the steady state. According to the work of Martinez-Garcia [13], since we have two state variables (k and a), there is a saddle point configuration if we have one negative characteristic root in the Jacobian matrix of our dynamical system linearized around the steady state $(k(q_0)^*, a(q_0)^*, p(q_0)^*)$.

Theorem 3. *For any given $q_0 > 0$, there exists a unique steady state $(k(q_0)^*, a(q_0)^*, p(q_0)^*)$, solution of (27)-(29), which correspond to a stable saddle-point configuration. There exists a unique optimal path converging to the NBGP characterized by theorem 1.*

Proof: see Appendix 5.3

An important implication of this theorem is the role played by q_0. The first part proves that the initial price of knowledge parametrizes the manifold of steady states for k, a and p. It's a classical feature of endogenous growth model, which was first in Lucas [12] paper's. As long as k_0 and a_0 are not the same for all the countries we have different starting point and also different asymptotic steady-states. An economy which is retarded from an other will reach a lower equilibrium even if the two have the same NBGP. Asymptotic position depends on initials conditions, inequalities remain present over time, this growth process don't suppress them.

For given initial conditions K(0) = k(0) = k_0 and A(0) = a(0) = a_0, generally it's impossible to find values for q_0 and p_0 in order to be located on the NBGP from the initial date. These values are adjustment variables, we need adjustments through the transitional dynamics. Since $(k_0, a_0) \neq (k^*(q_0), a^*(q_0))$ there will be adjustments, and p_0 and q_0 will be chosen to select the one-dimensional stable path which converges to the steady-state depending on q_0.

An important property is that, even if the steady state depends on the initial value q_0, the associated eigenvalues don't. The dynamic (i.e convergence speed) is the same for all initials endowments (a_0, k_0).

We can illustrate this with a figure. From the expression of k^* and a^* given by equations (45) and (47), in appendix 6.3, we have:

$$k^* = Z_1 a^* \tag{30}$$

With Z_1 also defined in the proof of theorem 2. All couple (k^*, a^*) satisfying (30) correspond to a common asymptotic NBGP but to different asymptotic steady states, depending on the initials conditions (a_0, k_0).

For given (a_0, k_0) the asymptotic path will converges to an asymptotic position on the curve defined by (30). As it's proved in appendix 6.3, this asymptotic position depends on the initial value q_0, and the relation between knowledge, a, and capital, k, is linear when $\epsilon > 1$. As it was mentioned, we cannot choose a q_0 in order to be directly located on the NBGP of our model. Some possible transition from (a_0, k_0) to (a^*, k^*) are illustrated by the arrow on figure 4. With these features, an economy beginning with a low level of capital and technological knowledge will converges to a lower steady-state than a better endowed economy.

This existence of a manifold of steady-state is a classical feature of endogenous growth literature, but it's a great difference with the results obtained by Acemoglu and Guerrieri. Indeed, in their paper they have an exogenous technical progress, their steady-state is unique and all countries will

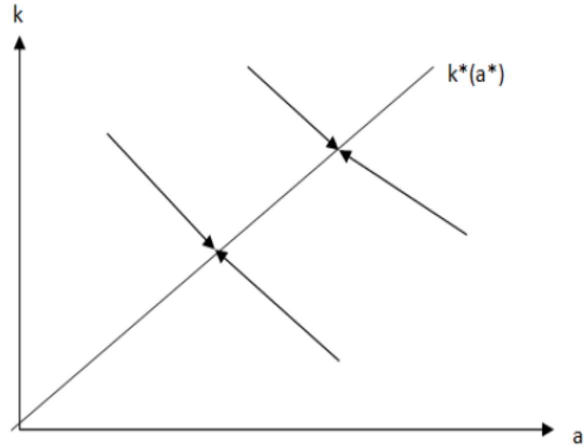


Figure 4: manifold of steady states ($\epsilon > 1$)

converge to the same NBGP and the same steady-state, no-matter their initial position. In terms of international macroeconomics, they have absolute convergence.

In our model, we are able to produce conditional convergence with identical strcuture for countries. In the classical Solow model, there is absolute convergence, all the countries converge to the same steady-state. If the countries differ in one parameter, for example in their saving rate, They will converge to different development level (different steady-states), there is conditional convergence. In these two cases, the initials conditions have no real impact.

In our model, we do not impose structural differences but according to the initials conditions (k_0, a_0), countries will converge to different level of wealth. There is conditional convergence. The initial development level matters, even if all countries reach the same NBGP, inequality remain constant across the development process. This property of endogenous growth models is more powerful than the properties of exogenous growth models. This feature is an important difference with Acemoglu and Guerrieri [1] because it allows for a development process analysis between countries.

4.2 Complementary intermediate inputs

In this section we will study the stability properties of the reverse case, i.e. when inputs are complements. We will use the same methodology, the stationarized system will be described by the following proposition :

Proposition 2. *Along a stationarized equilibrium, the dynamical system with respect to price, capital and knowledge (when inputs are complements) is the following*

$$\frac{\dot{p}}{p} = -\left[(1-\beta)(1-\gamma)\phi^{\frac{1}{\epsilon}}\left(\frac{L_2(k,a,p,q_0,N_0)}{K_2(k,a,p,q_0,N_0}\right)^{\beta} + g_P - \rho - \delta\right] \tag{31}$$

$$\frac{\dot{k}}{k} = \phi\left(\frac{L_2(k,a,p,q_0,N_0)}{K_2(k,a,p,q_0,N_0}\right)^{\beta}\frac{K_2}{k} - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k} \tag{32}$$

$$\frac{\dot{a}}{a} = z\left(1 - u(k,a,p,q_0,N_0)\right) - g_A - \eta \tag{33}$$

Proof: see appendix 6.4

In this configuration the convergence is different from the previous case, we need to impose some restrictions on the parameters to obtain a saddle-path. We still have the existence of a unique steady-state $(k(q_0)^*, a(q_0)^*, p(q_0)^*)$ for any $q_0 > 0$. We apply a similar methodology than in 4.1 to obtain the following theorem:

Theorem 4. *When $\epsilon < \alpha$, for any given $q_0 > 0$, there exists a unique steady-state $(k(q_0)^*, a(q_0)^*, p(q_0)^*)$, solution of (31)-(33) which correspond to a stable saddle-point configuration. When $\alpha < \epsilon < 1$ we have complete instability of the system.*

Proof: see appendix 6.5

When inputs are complements, there is a restriction to obtain a saddle-point, we need to have an elasticity of substitution smaller than the labor intensity. If ϵ is sufficiently low, it is optimal to invest in sector 2, as it was described in theorem 2, the growth rate in sector 2 are larger to compensate the lack of endogenous growth in this sector.

α corresponds to the labor intensity in sector 1, it also applies to the technical progress A_t. If this parameter is smaller than the elasticity of substitution it means that it is more efficient to invest in sector 1. In that case, inputs are not enough complementary to obtain a perfect compensation by redirecting capital and labor in sector 2. Even if there are not enough complementary, they are not substitutable so we cannot reach an optimal and stable path.

In this case, the model exhibits the same features of conditional convergence than in the reverse case. In appendix 6.5 we show that the manifold of steady states is described by the following equation:

$$k^* = X_3^{\frac{1}{1-\beta}} a^{*\frac{1}{1-\beta}}$$

The curvature is not the same than in the substitutable case, we have an exponential form of the relationship between knowledge and capital. It is represented in the graph below.

The main difference with the case $\epsilon > 1$ is the parabolic form of the relationship between knowledge, a, and capital, k. This form is implied by the fact that if a country has an high endowment of human capital (relatively to capital) at date 0, the complementarity between sectors impeach the plentiful

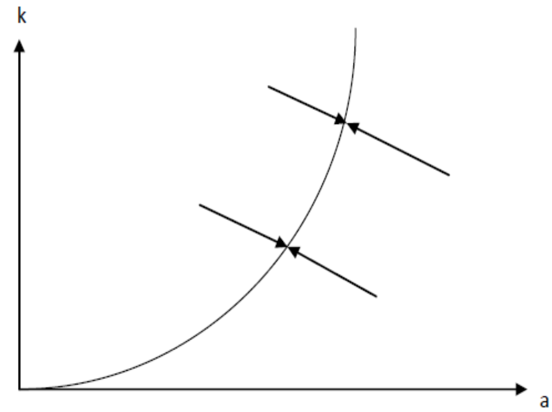


Figure 5: manifold of steady states ($\epsilon < 1$)

growth development. Indeed, only one sector is knowledge intensive, if a country invest intensively in this sector the complementarity of inputs lowers the production of final good. As we already mentioned it, capital deepening occurs more intensively in sector 2 to compensate the lack of endogenous growth. In the transition, the country needs to "sacrifice" a part of knowledge to fill the gap between our two sectors by capital accumulation.

5 Conclusion

We build a model of endogenous structural change, in a two sectors framework with a CES production function for the final good. The introduction of an elasticity of substitution determines the equilibrium we will reach (or not). In this purely theoretical work we have seen that if intermediates sectors are substitutable the Non Balanced Growth Path (NBGP) is always stable and the means of production are allocated more intensively in sector 1, the knowledge intensive sector. This result is quite obvious, as long as there is TFP improvement in only one sector, it will be efficient to transfer the majority of the production in this sector because of the substitutability.

However, the results are quite different when inputs are complementary. First, when $\epsilon < 1$ the trajectory is not always stable, to have a saddle-point we need $\epsilon < \alpha$, an elasticity of substitution smaller than the labor intensity of sector 1. Stability requires intermediates inputs sufficiently complementary. When it is the case, means of production are more intensively allocated in sector 2, we accumulate labor and capital in this sector to compensate the lack of endogenous growth, capital deepening is used to balanced with knowledge accumulation. When the complementarity condition is not reach (i.e. $1 > \epsilon > \alpha$), there is complete instability because the optimal way to allocated labor and capital between sectors is unclear. When inputs are not enough complementary it is still more efficient to accumulate means of production in sector 1 because endogenous growth dominates the complementarity effect, knowledge intensity overtakes combination necessity of the CES function, and in the long run the effect is unclear, we have complete instability.

Our endogenous growth framework also implies the manifold of steady-states feature. We have seen that the initial endowment of capital and knowledge (k_0, a_0) defines the level of wealth we reach, if we consider different countries with different initials endowments we will have conditional convergence: all countries will reach the same NBGP but not the same level of wealth. If inequalities are present at t=0, they remain along the development process. Each country will attains a particular steady-state which will correspond to a particular level of development, there are still inequalities.

Even for the manifold of steady-states feature, the value of epsilon matters. As we have seen when inputs are substitutable the relation between a and k is linear, but when they are complement the relation is exponential. For a given level of capital, the knowledge available will be lower if intermediate sectors are complementary. It is an important result, it means that country structure will define the trade-off between physical and human capital, in a "complementary world" there will be less human capital than in a "substitutable world". It points the importance of an empirical counterpart to this paper, in order to determines the value of the elasticity of substitution. It as been already done by Papageorgiou, Samm and Schulte [15] for the elasticity of substitution between clan and dirty sector in Acemoglu, Aghion, Bursztyn and Hemmous [2] paper's.

A way to extend this model would be to add a third sector, in order to try to reproduce the Kuznets facts on Agriculture, manufacturing and services. Or, elasticity of substitution could change across time, with the idea that substitutability between low-skilled work and high-skilled work could be different if we consider Industrial revolution period and current period. Such feature could maybe explains the hump-shape of labor share in manufacturing.

6 Appendix

6.1 Proof of theorem 1

Our aim is to derive all the growth rates of this economy along the non-balance growth path. We can begin by finding simple relations between the variables that we are interested on. But before starting it's important to recall that along the NBGP, all the growth rates are constant, so if we differentiate them, they equalize 0.

Along the NBGP we have $\dot{g_K} = 0$, it's equivalent to:

$$g_Y - g_K - n + \frac{1}{\theta} g_P + g_K = 0 \tag{34}$$

using the fact that $\dot{g_P} = 0$ give:

$$\begin{aligned} g_{K_1} &= \frac{\epsilon - 1}{\epsilon} g_{Y_1} + \frac{1}{\epsilon} g_Y & (35) \\ g_Q &= \rho - z + \eta & (36) \end{aligned}$$

Differentiating (4) gives this equality:

$$g_{Y_1} = \alpha g_A + \alpha n + (1 - \alpha) g_{K_1} \tag{37}$$

We Combine (9) and (10) and differentiate the result to obtain:

$$g_P + \frac{\epsilon - 1}{\epsilon} g_{Y_1} + \frac{1}{\epsilon} g_Y - g_A = g_Q \tag{38}$$

Now we differentiate equations (14) and (15):

$$\begin{aligned} \frac{\epsilon - 1}{\epsilon} g_{Y_1} - g_{K_1} &= \frac{\epsilon - 1}{\epsilon} g_{Y_2} - g_{K_2} & (39) \\ \frac{\epsilon - 1}{\epsilon} g_{Y_1} - n_1 &= \frac{\epsilon - 1}{\epsilon} g_{Y_1} - n_2 & (40) \end{aligned}$$

And finally, using equations (14), (5) and (15) we obtain the following equation:

$$g_{Y_2} = (1 - \beta)\epsilon g_{K_1} + \beta \epsilon n + (1 - \epsilon) g_{Y_1} \tag{41}$$

The 8 relations (34) – (41) between our growth rates are not enough to determine explicitly their value, we need some more tools. So to determine the relationship between the growth rate of K and K_1, L and L_1, Y and Y_1 we use the same methodology than in Acemoglu and Guerierri. We will use the functional form of $\kappa(t)$ and $\lambda(t)$, and the maximized value of output $\Phi(K(t), t)$.

Now our aim is to derive growth rate of consumption. With this we will be able to use the fact that along the NBGP the growth rate of consumption is constant, so if we differentiate it we obtain $\dot{g_c} = 0$.

We write the partial Hamiltonian in maximized value:

$$\mathbb{H}(c,K,P) \quad = \quad N\tfrac{c^{1-\theta}}{1-\theta}e^{-\rho t} + P\Big[\Phi(K(t),t) - \delta K - Nce^{-\rho t}\Big]$$

The optimization conditions give:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\Big[(1-\alpha)\psi(t)A^{\alpha}\lambda^{\alpha}\kappa^{-\alpha}L^{\alpha}K^{-\alpha} - \delta - \rho\Big] \tag{42}$$

Along the NBGP $\dot{g_c} = 0$, we differentiate (42) and equalize it to 0:

$$\frac{\dot{\psi}(t)}{\psi(t)} + \alpha\frac{\dot{A}}{A} + \alpha\frac{\dot{\lambda}(t)}{\lambda(t)} - \alpha\frac{\dot{\kappa}(t)}{\kappa(t)} + \alpha\frac{\dot{L}(t)}{L(t)} - \alpha\frac{\dot{K}(t)}{K(t)} = 0$$

We can differentiate all the components of this equality and express them in terms of $\frac{\dot{\kappa}(t)}{\kappa(t)}$ (except for A) such that:

$$\begin{aligned}
\frac{\dot{\psi}(t)}{\psi(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left(\frac{\epsilon(1-\alpha)}{[\Delta\kappa + (1-\alpha)](1-\epsilon)}\right) \\
\frac{\dot{\lambda}(t)}{\lambda(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1} \\
\frac{\dot{K}(t)}{K(t)} &= \frac{\dot{\kappa}(t)}{\kappa(t)}\left[\frac{(1-\alpha)\epsilon}{\alpha(1-\epsilon)[\Delta\kappa + 1 - \alpha]} + \left(\frac{(1-\beta)\alpha}{(1-\alpha)\beta} + \frac{1-\kappa}{\kappa}\right)^{-1}\right] + \Big(z(1-u) - \eta + n\Big)
\end{aligned}$$

Where $\Delta = \alpha - \beta$

Substituting all these components into the previous equality $\dot{g_c} = 0$ allows to rewrite the growth rate of $\kappa(t)$ as follows:

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = \frac{[\Delta\kappa + (1-\alpha)](1-\kappa)}{(1-\alpha)(\epsilon\Delta - \alpha) + \Delta\kappa(\alpha(\epsilon-1) - \epsilon)}(1-\epsilon)\alpha\beta(z(1-u) - \eta) \tag{43}$$

$$\frac{\dot{\kappa}(t)}{\kappa(t)} = G(\kappa(t)).(1-\epsilon)\alpha\beta(z(1-u) - \eta) \tag{44}$$

We've got $G(0) < 0$, $G(1) = 0$ and $G'(\kappa) > 0$ for any κ.
Close to the infinity, the growth rate of κ must be equal to 0, it's obvious according to the previous observations that the asymptotic value of κ is 1. When $t \to \infty$, $\kappa(t) \to \kappa* = 1$. Using this asymptotic value into λ and ψ give the following results: $\lambda \to \lambda* = 1$ and $\psi \to \psi* = \gamma^{\frac{\epsilon}{\epsilon-1}}$.

Now, using equations (34) and the previous results we can state that $g_K = g_{K_1}$ and $n = n_1$. And using the maximized value of the output $\Phi(K(t),t)$ as in equation (22) we obtain $Y(t) = \psi.Y_1(t)$, which give $g_{Y_1} = g_Y$.

Now we can replace all these equalities into the equations $(38)-(41)$ to obtain the explicit values of our growth rates. from (35) we have $g_K = g_Y$ Using $(34), (36), (38)$, this equality and $g_A = z(1-u) - \eta$ we obtain:

$$u^* = \frac{1}{z}\Big[z - \eta - \frac{1}{\theta}(n - \rho + z - \eta)\Big]$$

Using $\frac{\dot{\kappa}}{\kappa} = 0$ along the NBGP and the expression of u, we can extract the explicit expression of g_K:

$$g_K = n + \frac{1}{\theta}\bigg(n - \rho + z - \eta\bigg)$$

Using this we can give all the growth rates of our economy, along the NBGP. *QED*.

6.2 Proof of proposition 1

To obtain the first 2 dynamical equations we use the maximized output: $Y(t) = \Phi(K(t), t)$ into equations (23) and (26), we also add (4) into equation (23) and then, by reorganizing, we obtain (27) and (28) but in their non-stationarized form.

The next step is to prove that $L_1 = L_1(k, a, p, q_0, N_0)$, $K_1 = K_1(k, a, p, q_0, N_0)$ and $u = u(k, a, p, q_0, N_0)$, and to compute the derivatives. For this purpose we will use the implicit function theorem.

From (14) and (15) we derive:

$$\begin{aligned}
(14) \quad &\Leftrightarrow \quad \frac{\gamma(1-\alpha)}{(1-\gamma)(1-\beta)} \frac{[(L_1 A)^\alpha K_1^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{K_1} = \frac{[(L-L_1)^\beta (K-K_1)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{K-K_1} \\
(15) \quad &\Leftrightarrow \quad \frac{\gamma\alpha}{(1-\gamma)\beta} \frac{[(L_1 A)^\alpha K_1^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{L_1} = \frac{[(L-L_1)^\beta (K-K_1)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{L-L_1} \\
\frac{(14)}{(15)} \quad &\Leftrightarrow \quad \frac{B_1}{B_2} L_1 (K - K_1) = K_1 (L - L_1)
\end{aligned}$$

with $B_1 = \frac{\gamma(1-\alpha)}{(1-\gamma)(1-\beta)}$ and $B_2 = \frac{\gamma\alpha}{(1-\gamma)\beta}$.

We characterize:

$$\begin{aligned}
\omega_1 &= \frac{B_1}{B_2} L_1 (K - K_1) - K_1 (L - L_1) = 0 \\
\omega_2 &= B_1 \frac{[(L_1 A)^\alpha K_1^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{K_1} - \frac{[(L-L_1)^\beta (K-K_1)^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{K-K_1} = 0 \\
&\Rightarrow \Omega(K_1, L_1, K, A, L) = 0
\end{aligned}$$

We assume $\frac{\partial \Omega}{\partial (K_1, L_1)} \neq 0$.

If $J_1 = \begin{pmatrix} \frac{\partial \omega_1}{\partial K_1} & \frac{\partial \omega_1}{\partial L_1} \\ \frac{\partial \omega_2}{\partial K_1} & \frac{\partial \omega_2}{\partial L_1} \end{pmatrix}$ is non-singular, there exists a unique $K_1 = K_1(K, A, L)$ and $L_1 = L_1(K, A, L)$, with:

$$\frac{\partial (K_1, L_1)}{\partial (K, A, L)} = J_1^{-1} \begin{pmatrix} \frac{\partial \omega_1}{\partial K} & \frac{\partial \omega_1}{\partial A} & \frac{\partial \omega_1}{\partial L} \\ \frac{\partial \omega_2}{\partial K} & \frac{\partial \omega_2}{\partial A} & \frac{\partial \omega_2}{\partial L} \end{pmatrix}$$

After some tedious computations we obtain the following matrix:

$$\begin{pmatrix} \frac{\partial K_1}{\partial K} & \frac{\partial K_1}{\partial A} & \frac{\partial K_1}{\partial L} \\ \frac{\partial L_1}{\partial K} & \frac{\partial L_1}{\partial A} & \frac{\partial L_1}{\partial L} \end{pmatrix}$$

such that:

$$\begin{aligned}
\frac{\partial K_1}{\partial K} &= \frac{K_1}{\Upsilon}\Bigg[(\alpha-\beta)(\epsilon-1)(L-L_1)-L\Bigg] \\
\frac{\partial K_1}{\partial A} &= -\frac{1}{\Upsilon}\alpha(\epsilon-1)(K-K_1)K_1 L\frac{1}{A} \\
\frac{\partial K_1}{\partial L} &= -\frac{K_1}{\Upsilon}(\epsilon-1)(\alpha-\beta)(K-K_1) \\
\frac{\partial L_1}{\partial K} &= \frac{1}{\Upsilon}\frac{B_1 L_1}{B_2 K_1}L_1(\alpha-\beta)(\epsilon-1)(K-K_1) \\
\frac{\partial L_1}{\partial A} &= -\frac{1}{\Upsilon}\frac{B_1 L_1}{B_2 K_1}\alpha(\epsilon-1)KL_1(K-K_1)\frac{1}{A} \\
\frac{\partial L_1}{\partial L} &= \frac{L_1}{\Upsilon}\Bigg[(\alpha-\beta)(\epsilon-1)(K-K_1)-K\Bigg]
\end{aligned}$$

with $\Upsilon = \Upsilon(L, K, L_1, K_1) = \Bigg[(\epsilon-1)(\alpha-\beta)(\frac{K_1}{K} - \frac{L_1}{L}) - 1\Bigg]LK$. It has no economic sense, it's just to lighten the notations.

The objective is to prove that $K_1 = K_1(K, A, P, Q, N)$ and $L_1 = L_1(K, A, P, Q, N)$. For the moment we have $K_1 = K_1(K, A, L)$ and $L_1 = L_1(K, A, L)$. We use market clearing conditions to assess $L = N-(1-u)N_1$ in order to obtain $K_1 = K_1(K, A, N-(1-u)N_1)$ and $L_1 = L_1(K, A, N-(1-u)N_1)$.

The first step is to prove that $N_1 = N_1(K, A, u, N)$, and then to prove $u = u(K, A, P, Q, N)$. When these 2 will be prove, we will have

$$K_1 = K_1(K, A, N-(1-u(K, A, P, Q, N)N_1(K, A, u(K, A, P, Q, N))) \Leftrightarrow K_1 = \breve{K}_1(K, A, P, Q, N)$$

As previously we will use the implicit function theorem to compute the derivatives of N_1 with respect to K, A, u and N:

We have $H \equiv uN_1 - L_1(K, A, N-(1-u)N_1) = 0$

Assumption: $\frac{\partial H}{\partial N_1} \neq 0$

$$\frac{\partial N_1}{\partial K} = -\frac{\frac{\partial H}{\partial K}}{\frac{\partial H}{\partial N_1}} \quad ; \quad \frac{\partial N_1}{\partial A} = -\frac{\frac{\partial H}{\partial A}}{\frac{\partial H}{\partial N_1}} \quad ; \quad \frac{\partial N_1}{\partial N} = -\frac{\frac{\partial H}{\partial N}}{\frac{\partial H}{\partial N_1}} \quad ; \quad \frac{\partial N_1}{\partial u} = -\frac{\frac{\partial H}{\partial u}}{\frac{\partial H}{\partial N_1}}$$

Computing these derivatives gives:

$$\begin{aligned}
\frac{\partial H}{\partial K} &= -\frac{\partial L_1}{\partial K} \\
\frac{\partial H}{\partial A} &= -\frac{\partial L_1}{\partial A} \\
\frac{\partial H}{\partial N} &= -\frac{\partial L_1}{\partial L} \\
\frac{\partial H}{\partial u} &= N_1 - N_1\frac{\partial L_1}{\partial L} \\
\frac{\partial H}{\partial N_1} &= u + (1-u)\frac{\partial L_1}{\partial L}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial N_1}{\partial K} &= \frac{\frac{\partial L_1}{\partial K}}{u + (1-u)\frac{\partial L_1}{\partial L}} \\
\frac{\partial N_1}{\partial A} &= \frac{\frac{\partial L_1}{\partial A}}{u + (1-u)\frac{\partial L_1}{\partial L}} \\
\frac{\partial N_1}{\partial N} &= \frac{\frac{\partial L_1}{\partial L}}{u + (1-u)\frac{\partial L_1}{\partial L}} \\
\frac{\partial N_1}{\partial u} &= -\frac{N_1(1-\frac{\partial L_1}{\partial L})}{u + (1-u)\frac{\partial L_1}{\partial L}}
\end{aligned}$$

We've seen it's possible to express all the derivatives with respect to the derivatives computed previously. We have proved that $N_1 = N_1(K, A, u, N)$. Now, we will use equation 9 to express u as a function of K, A, P, Q and N.

We will proceed by introducing equation (9) such that:

$$G = \frac{\alpha\gamma}{zAu} PY(K, A, u, N)^{\frac{1}{\epsilon}} Y_1(K, A, u, N)^{\frac{\epsilon-1}{\epsilon}} - Q = 0$$

Under the assumption $\frac{\partial G}{\partial u} \neq 0$ there exists a unique u = u(K,A,P,Q,N) such that:

$$\frac{\partial u}{\partial K} = -\frac{\frac{\partial G}{\partial K}}{\frac{\partial G}{\partial u}} \quad ; \quad \frac{\partial u}{\partial A} = -\frac{\frac{\partial G}{\partial A}}{\frac{\partial G}{\partial u}} \quad ; \quad \frac{\partial u}{\partial P} = -\frac{\frac{\partial G}{\partial P}}{\frac{\partial G}{\partial u}} \quad ; \quad \frac{\partial u}{\partial Q} = -\frac{\frac{\partial G}{\partial Q}}{\frac{\partial G}{\partial u}} \quad ; \quad \frac{\partial u}{\partial N} = -\frac{\frac{\partial G}{\partial N}}{\frac{\partial G}{\partial u}}$$

with:

$$\begin{aligned}
\frac{\partial G}{\partial K} &= Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial K}+\frac{1-\epsilon}{\epsilon}\frac{1}{Y_1}\partial Y_1\partial K\right]\\
\frac{\partial G}{\partial A} &= Q\left[-\frac{1}{A}+\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial A}+\frac{1-\epsilon}{\epsilon}\frac{1}{Y_1}\partial Y_1\partial A\right]\\
\frac{\partial G}{\partial P} &= \frac{Q}{P}\\
\frac{\partial G}{\partial u} &= Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial u}+\frac{1-\epsilon}{\epsilon}\frac{1}{Y_1}\partial Y_1\partial u-\frac{1}{u}\right]\\
\frac{\partial G}{\partial Q} &= -1\\
\frac{\partial G}{\partial N} &= Q\left[\frac{1}{\epsilon}\frac{1}{Y}\frac{\partial Y}{\partial N}+\frac{1-\epsilon}{\epsilon}\frac{1}{Y_1}\partial Y_1\partial N\right]
\end{aligned}$$

With some computations it's possible to express all the derivatives with respect to the previous ones, we give an example for K:

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= \left(\gamma\frac{\partial Y_1}{\partial K}Y_1^{-\frac{1}{\epsilon}}+(1-\gamma)\frac{\partial Y_2}{\partial K}Y_2^{-\frac{1}{\epsilon}}\right)Y^{\frac{1}{\epsilon}}\\
&\Rightarrow\ \frac{\partial Y_1}{\partial K}=Y_1\left(\alpha\frac{1}{L_1}\frac{\partial \tilde{L_1}}{\partial K}+(1-\alpha)\frac{1}{K_1}\frac{\partial \tilde{K_1}}{\partial K}\right)\\
&\Rightarrow\ \frac{\partial Y_2}{\partial K}=-Y_2\left(\beta\frac{1}{L-L_1}\frac{\partial \tilde{L_1}}{\partial K}+(1-\beta)\frac{1}{K-K_1}\frac{\partial \tilde{K_1}}{\partial K}\right)\\
&\qquad\Rightarrow\frac{\partial \tilde{L_1}}{\partial K}=\frac{\partial L_1}{\partial K}\left(\frac{u}{u+(1-u)\frac{\partial L_1}{\partial L}}\right)\\
&\qquad\Rightarrow\frac{\tilde{K_1}}{\partial K}=\frac{\partial K_1}{\partial K}-(1-u)\frac{\partial K_1}{\partial L}\frac{\frac{\partial L_1}{\partial K}}{u+(1-u)\frac{\partial L_1}{\partial L}}
\end{aligned}$$

With the tilde subscript used to express $K_1 = K_1(K,A,N-(1-u)N_1(K,A,u,N)) = \tilde{K_1}(K,A,u,N)$ and the same for L_1.

With this decomposition we have expressed all the derivatives included into $\frac{\partial G}{\partial K}$ in terms of the derivatives computed in the first part of the proof. We can repeat this for all the derivatives of G to obtain what we want: the derivatives of u with respect to K, A, P, Q and N.

We have proved that $N_1 = N_1(K,A,u,N)$ and $u = u(K,A,P,Q,N)$, we can now rewrite N_1 as follow: $N_1 = N_1(K, A, u(K, A, P, Q, N), N) = \tilde{N_1}(K, A, P, Q, N)$ such that:

$$\begin{aligned}
\frac{\partial \tilde{N_1}}{\partial K} &= \frac{\partial N_1}{\partial K} + \frac{\partial N_1}{\partial u}\frac{\partial u}{\partial K} \\
\frac{\partial \tilde{N_1}}{\partial A} &= \frac{\partial N_1}{\partial A} + \frac{\partial N_1}{\partial u}\frac{\partial u}{\partial A} \\
\frac{\partial \tilde{N_1}}{\partial P} &= \frac{\partial N_1}{\partial u}\frac{\partial u}{\partial P} \\
\frac{\partial \tilde{N_1}}{\partial Q} &= \frac{\partial N_1}{\partial u}\frac{\partial u}{\partial Q} \\
\frac{\partial \tilde{N_1}}{\partial N} &= \frac{\partial N_1}{\partial N} + \frac{\partial N_1}{\partial u}\frac{\partial u}{\partial N}
\end{aligned}$$

Using these results additionally to $K_1 = \tilde{K_1}(K, A, N - (1 - u(K, A, P, Q, N)N_1(K, A, P, Q, N))$ and $L_1 = \tilde{L_1}(K, A, N - (1 - u(K, A, P, Q, N)N_1(K, A, P, Q, N)$ we obtain: (we keep the gross form of our derivatives for clarity)

$$\begin{aligned}
\frac{\partial \tilde{K_1}}{\partial K} &= \frac{\partial K_1}{\partial K} + \frac{\partial K_1}{\partial L}\left[\frac{\partial u}{\partial K}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial K}\right] \\
\frac{\partial \tilde{K_1}}{\partial A} &= \frac{\partial K_1}{\partial A} + \frac{\partial K_1}{\partial L}\left[\frac{\partial u}{\partial A}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial A}\right] \\
\frac{\partial \tilde{K_1}}{\partial P} &= \frac{\partial K_1}{\partial L}\left[\frac{\partial u}{\partial P}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial P}\right] \\
\frac{\partial \tilde{K_1}}{\partial Q} &= \frac{\partial K_1}{\partial L}\left[\frac{\partial u}{\partial Q}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial Q}\right] \\
\frac{\partial \tilde{K_1}}{\partial N} &= \frac{\partial K_1}{\partial N} + \frac{\partial K_1}{\partial L}\left[\frac{\partial u}{\partial N}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial N}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \tilde{L_1}}{\partial K} &= \frac{\partial L_1}{\partial K} + \frac{\partial L_1}{\partial L}\left[\frac{\partial u}{\partial K}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial K}\right] \\
\frac{\partial \tilde{L_1}}{\partial A} &= \frac{\partial L_1}{\partial A} + \frac{\partial L_1}{\partial L}\left[\frac{\partial u}{\partial A}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial A}\right] \\
\frac{\partial \tilde{L_1}}{\partial P} &= \frac{\partial L_1}{\partial L}\left[\frac{\partial u}{\partial P}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial P}\right] \\
\frac{\partial \tilde{L_1}}{\partial Q} &= \frac{\partial L_1}{\partial L}\left[\frac{\partial u}{\partial Q}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial Q}\right] \\
\frac{\partial \tilde{L_1}}{\partial N} &= \frac{\partial K_1}{\partial N} + \frac{\partial L_1}{\partial L}\left[\frac{\partial u}{\partial N}\tilde{N_1} - (1-u)\frac{\partial \tilde{N_1}}{\partial N}\right]
\end{aligned}$$

We have proved that K_1 and L_1 are function of K, A, P, Q and N, and we have all the derivatives of our main variables. Using the property that a CES function is homogeneous of degree 1 as all the other functions of our model we state:

$$\begin{aligned} K_1 \quad (K, A, P, Q, N) &= K_1(k, a, p, q_0, N_0) \\ L_1 \quad (K, A, P, Q, N) &= L_1(k, a, p, q_0, N_0) \\ u \quad (K, A, P, Q, N) &= u(k, a, p, q_0, N_0) \end{aligned}$$

QED

6.3 Proof of theorem 3

First of all we need to compute the unique steady-state (k^*, a^*, p^*). We considered a close form of the dynamical system in proposition 1.

$$(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}a^{\alpha}\lambda^{\alpha}\kappa^{-\alpha}l^{\alpha}k^{-\alpha} = \rho + \delta - g_P \tag{45}$$

$$\psi a^{\alpha}\lambda^{\alpha}\kappa^{1-\alpha}l^{\alpha}k^{-\alpha} = \delta + g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k} \tag{46}$$

$$z(1-u) = g_A + \eta \tag{47}$$

From (41) we have $u = \frac{z - g_A - \eta}{z} \equiv u^*$
We can use this to obtain steady state value l^*, indeed as we have shown, at steady state $\lambda^* = 1 \Leftrightarrow L_1 = L$, and $N_1 = N$. Recalling $L = N - (1-u)N_1$ at the stationarized steady state we have $l^* = u^* N_0$.
We also apply the following steady state values : $\kappa^* = 1$ and $\psi^* = \gamma^{\frac{\epsilon-1}{\epsilon}}$ to equations (39) and (40) to obtain:

$$k = Z_1 a \tag{48}$$

$$p = Z_2 a^{-\theta} \tag{49}$$

$$\textit{with } \; Z_1 = \left(\frac{(1-\alpha)\gamma^{\frac{\epsilon-1}{\epsilon}}}{\rho+\delta-g_P}\right)^{\frac{1}{\alpha}} l^* \; \textit{ and } \; Z_2 = \left(\frac{(1-\alpha)N_0}{\{(1-\alpha)(\delta+g_K) - (\rho+\delta-g_P)\}Z_1}\right)^{\theta}$$

In order to find a^* we used (9) evaluated at steady state such that:

$$p\gamma\psi^{\frac{1}{\epsilon}}\alpha(\lambda^{\alpha}l^{\alpha}a^{\alpha}\kappa^{1-\alpha}k^{1-\alpha}) = q_0 z u a$$

And we obtain:

$$a^* = \left(\frac{Z_2 Z_1^{1-\alpha}\gamma^{\frac{\epsilon-1}{\epsilon}}\alpha l^{*\alpha}}{q_0 z u^*}\right)^{\frac{1}{\theta}} \tag{50}$$

Introducing this formula into k and p proves the existence of a unique steady state (k^*, a^*, p^*) which depends on the initial value q_0.

We already have all the partial derivatives of L_1, K_1 and u with respect to K, A, P, Q and N. We can linearize our dynamical systel around the steady-state (k^*, a^*, p^*).

Let us now consider the dynamical system

$$\begin{aligned}
\dot{p} &= -p\left\{(1-\alpha)\gamma\psi^{\frac{1}{\epsilon}}a^{\alpha}\left(\frac{L_1(k,a,p,q_0,N_0)}{K_1(k,a,p,q_0,N_0}\right)^{\alpha} + g_P - \rho - \delta\right\} \equiv \mathcal{F}(p,k,a) \\
\dot{k} &= k\left\{\psi a^{\alpha}\left(\frac{L_1(k,a,p,q_0,N_0)}{K_1(k,a,p,q_0,N_0}\right)^{\alpha}\kappa - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k}\right\} \equiv \mathcal{G}(p,k,a) \\
\dot{a} &= a\left\{z\Big(1-u(k,a,p,q_0,N_0)\Big) - g_A - \eta\right\} \equiv \mathcal{H}(p,k,a)
\end{aligned}$$

The linearization around the steady state yields the following Jacobian matrix:

$$\mathcal{J} = \begin{pmatrix} \mathcal{F}_1(p^*,k^*,a^*) & \mathcal{F}_2(p^*,k^*,a^*) & \mathcal{F}_3(p^*,k^*,a^*) \\ \mathcal{G}_1(p^*,k^*,a^*) & \mathcal{G}_2(p^*,k^*,a^*) & \mathcal{G}_3(p^*,k^*,a^*) \\ \mathcal{H}_1(p^*,k^*,a^*) & \mathcal{H}_2(p^*,k^*,a^*) & \mathcal{H}_3(p^*,k^*,a^*) \end{pmatrix}$$

with

$$\begin{aligned}
\mathcal{F}_1(p^*,k^*,a^*) &= -(\rho+\delta-g_P)\tfrac{\alpha}{1-\alpha} < 0 \\
\mathcal{F}_2(p^*,k^*,a^*) &= 0 \\
\mathcal{F}_3(p^*,k^*,a^*) &= (\rho+\delta-g_P)\alpha\tfrac{p^*}{a^*(1-\alpha)} > 0 \\
\mathcal{G}_1(p^*,k^*,a^*) &= (\rho+\delta-g_P)\tfrac{k^*}{p^*}\tfrac{\alpha}{(1-\alpha)^2} + \tfrac{1}{\theta}N_0 p^{*\frac{-1-\theta}{\theta}} > 0 \\
\mathcal{G}_2(p^*,k^*,a^*) &= (\rho+\delta-g_P)\tfrac{1}{1-\alpha} - \delta - g_K > 0 \\
\mathcal{G}_3(p^*,k^*,a^*) &= (\rho+\delta-g_P)\tfrac{1}{(1-\alpha)^2}\tfrac{k^*}{a^*} > 0 \\
\mathcal{H}_1(p^*,k^*,a^*) &= -\tfrac{a^* z u^*}{p^*(1-\alpha)} < 0 \\
\mathcal{H}_2(p^*,k^*,a^*) &= -\tfrac{a^* z u^*}{k^*} < 0 \\
\mathcal{H}_3(p^*,k^*,a^*) &= z(1-u^*) - g_A - \eta\tfrac{z u^*}{1-\alpha} > 0
\end{aligned}$$

We then derive the characteristic polynomial

$$\mathcal{Q}(\lambda) = \lambda^3 - \lambda^2\mathcal{T} + \lambda\mathcal{S} - \mathcal{D} \tag{51}$$

with

$$\begin{aligned}
\mathcal{T} &= \mathcal{F}_1(p^*,k^*,a^*) + \mathcal{G}_2(p^*,k^*,a^*) + \mathcal{H}_3(p^*,k^*,a^*) \\
\mathcal{S} &= \mathcal{F}_1(p^*,k^*,a^*)\mathcal{G}_2(p^*,k^*,a^*) - \mathcal{G}_1(p^*,k^*,a^*)\mathcal{F}_2(p^*,k^*,a^*) \\
&+ \mathcal{G}_2(p^*,k^*,a^*)\mathcal{H}_3(p^*,k^*,a^*) - \mathcal{H}_2(p^*,k^*,a^*)\mathcal{G}_3(p^*,k^*,a^*) \\
&+ \mathcal{F}_1(p^*,k^*,a^*)\mathcal{H}_3(p^*,k^*,a^*) - \mathcal{H}_1(p^*,k^*,a^*)\mathcal{F}_3(p^*,k^*,a^*) \\
\mathcal{D} &= \mathcal{F}_1(p^*,k^*,a^*)\left[\mathcal{G}_2(p^*,k^*,a^*)\mathcal{H}_3(p^*,k^*,a^*) - \mathcal{G}_3(p^*,k^*,a^*)\mathcal{H}_2(p^*,k^*,a^*)\right] \\
&+ \mathcal{F}_3(p^*,k^*,a^*)\left[\mathcal{G}_1(p^*,k^*,a^*)\mathcal{H}_2(p^*,k^*,a^*) - \mathcal{G}_2(p^*,k^*,a^*)\mathcal{H}_1(p^*,k^*,a^*)\right]
\end{aligned}$$

We compute the different elements in the determinant:

$$
\begin{aligned}
\mathcal{G}_2\mathcal{H}_3 - \mathcal{G}_3\mathcal{H}_2 &= [(\rho+\delta-g_P)\frac{1}{1-\alpha} - \delta - g_K].[z(1-u^* - g_A - \eta + \frac{u^* z}{1-\alpha}] \\
&+ [(\rho+\delta-g_P)\frac{1}{(1-\alpha)^2}\frac{k^*}{a^*}].[\frac{a^* u^* z}{k^*}] > 0 \\
\mathcal{G}_1\mathcal{H}_2 - \mathcal{G}_2\mathcal{H}_1 &= \frac{a^* u^* z}{(1-\alpha)p^*}(\rho - g_P - g_K - \frac{1}{\theta}N_0 {p^*}^{\frac{-1}{\theta}}\frac{1-\alpha}{k^*} < 0
\end{aligned}
$$

It's straightforward to see that the determinant is negative: $D < 0$.

Now we are interested on Trace computation:

$$\mathcal{T} = \rho + \frac{u^* z}{1-\alpha} - g_p - g_K > 0$$

Since $\mathcal{T} > 0$ and $\mathcal{D} < 0$, we conclude that any given steady state $(p^*(q_0), k^*(q_0), a^*(q_0))$ on the manifold, the local stability properties are the same.These are the conditions to have a saddle-point stable steady-state. Therefore, for any given $q_0 > 0$, there exists a unique $p_0 > 0$ such that the unique converging path is on the stable manifold of dimension one. Along this converging path all the variables are bounded and the transversality conditions are satisfied: this converging path is the unique optimal solution. *QED*

6.4 Proof of proposition 2

We will use exactly the same methodology as in 6.2, the only difference is that now we will focus on sector 2, instead of sector 1.

$$
\begin{aligned}
\omega_1' &= \frac{B_1}{B_2}(L-L_2)K_2 - (K-K_2)L_2 = 0 \\
\omega_2' &= B_1\frac{[((L-L_2)A)^\alpha (K-K_2)^{1-\alpha}]^{\frac{\epsilon-1}{\epsilon}}}{K_1} - \frac{[L_2^\beta K_2^{1-\beta}]^{\frac{\epsilon-1}{\epsilon}}}{K-K_1} = 0 \\
&\Rightarrow \Omega'(K_1, L_1, K, A, L) = 0
\end{aligned}
$$

We assume $\frac{\partial \Omega'}{\partial (K_2, L_2)} \neq 0$.

If $J_2 = \begin{pmatrix} \frac{\partial \omega_1'}{\partial K_2} & \frac{\partial \omega_1'}{\partial L_2} \\ \frac{\partial \omega_2'}{\partial K_2} & \frac{\partial \omega_2'}{\partial L_2} \end{pmatrix}$ is non-singular, there exists a unique $K_2 = K_2(K, A, L)$ and $L_2 = L_2(K, A, L)$, with:

$\frac{\partial (K_2, L_2)}{\partial (K, A, L)} = J_2^{-1} \begin{pmatrix} \frac{\partial \omega_1'}{\partial K} & \frac{\partial \omega_1'}{\partial A} & \frac{\partial \omega_1'}{\partial L} \\ \frac{\partial \omega_2'}{\partial K} & \frac{\partial \omega_2'}{\partial A} & \frac{\partial \omega_2'}{\partial L} \end{pmatrix}$

We obtain:

$$\begin{aligned}
\frac{\partial K_2}{\partial K} &= -\frac{1}{\Lambda}\left[L_2(B_1\frac{\alpha(\epsilon-1)}{\epsilon(L-L_2)}+\frac{\beta(\epsilon-1)}{\epsilon L_2})+\frac{(1-\alpha)(\epsilon-1)-\epsilon}{\epsilon(K-K_2)}B_1(\frac{B_1}{B_2}K_2+K-K_2)\right]\\
\frac{\partial K_2}{\partial A} &= -\frac{1}{\Lambda}\left[(\frac{B_1}{B_2}K_2+K-K_2)(B_1\frac{\alpha(\epsilon-1)}{\epsilon A})\right]\\
\frac{\partial K_2}{\partial L} &= \frac{1}{\Lambda}\left[\frac{B_1}{B_2}K_2(B_1\frac{\alpha(\epsilon-1)}{\epsilon(L-L_2)}-\frac{\beta(\epsilon-1)}{\epsilon L_2})+\frac{(\alpha)(\epsilon-1)B_1}{\epsilon(L-L_2)}\right]\\
\frac{\partial L_2}{\partial K} &= \frac{1}{\Lambda}\left[L_2(B_1\frac{(1-\alpha)(\epsilon-1)-\epsilon}{\epsilon(K-K_2)}+\frac{(1-\beta)(\epsilon-1)-\epsilon}{\epsilon K_2})+\frac{(1-\alpha)(\epsilon-1)-\epsilon}{\epsilon(K-K_2)}B_1(\frac{B_1}{B_2}(L-L_2)+L_2)\right]\\
\frac{\partial L_1}{\partial A} &= \frac{1}{\Lambda}\left[(\frac{B_1}{B_2}(L-L_2)+L_2)(B_1\frac{\alpha(\epsilon-1)}{\epsilon A})\right]\\
\frac{\partial L_1}{\partial L} &= -\frac{1}{\Lambda}\left[\frac{B_1}{B_2}K_2(B_1\frac{(1-\alpha)(\epsilon-1)-\epsilon}{\epsilon(K-K_2)}+\frac{(1-\beta)(\epsilon-1)-\epsilon}{\epsilon K_2})+B_1\frac{\alpha)(\epsilon-1)}{\epsilon(L-L_2)}(\frac{B_1}{B_2}(L-L_2)+L_2)\right]
\end{aligned}$$

with

$$\Lambda=\frac{B_1}{B_2}\left(B_1\frac{\alpha(\epsilon-1)}{\epsilon}+\frac{(1-\beta)(\epsilon-1)-\epsilon}{\epsilon}\right)+\frac{\beta(\epsilon-1)}{\epsilon}+B_1\frac{(1-\alpha)(\epsilon-1)-\epsilon}{\epsilon}-\frac{B_1L_2}{(L-L_2)\epsilon}-\frac{K-K_2}{K_2\epsilon}$$

As in 6.2 we use implicit function theorem to compute derivative with respect to K, A, P, Q and N to show that L_2 and K_2 are 2 functions of these 5 elements.

We end up with

$$\begin{aligned}
K_2 &\quad (K,A,P,Q,N)=K_2(k,a,p,q_0,N_0)\\
L_2 &\quad (K,A,P,Q,N)=L_2(k,a,p,q_0,N_0)\\
u &\quad (K,A,P,Q,N)=u(k,a,p,q_0,N_0)
\end{aligned}$$

6.5 Proof of theorem 4

We first compute the stationarized steady state using a closed form of the equations in proposition 2:

$$(1-\gamma)(1-\beta)\phi^{\frac{1}{\epsilon}}(1-\lambda)^{\beta}(1-\kappa)^{-\beta}l^{\beta}k^{-\beta} = \rho+\delta-g_P \tag{52}$$

$$\phi(1-\lambda)^{\beta}(1-\kappa)^{1-\beta}l^{\beta}k^{-\beta} = \delta+\frac{N_0p^{-\frac{1}{\theta}}}{k} \tag{53}$$

$$z(1-u) = g_A+\eta \tag{54}$$

We solve this system and we obtain

$$k^*=X_1 \qquad p^*=X_2 \qquad a^*=\frac{X_1^{1-\beta}}{X_3}$$

$$With \qquad X_1\equiv\left(\frac{(1-\gamma)^{\frac{\epsilon}{\epsilon-1}}(1-\beta)}{\rho+\delta-g_P}\right)^{\frac{1}{\beta}}N_0 \qquad ; \qquad X_2\equiv\left(\frac{(1-\gamma)^{\frac{\epsilon}{\epsilon-1}}N_0^{\beta}X_1^{1-\beta}-\delta}{N_0}\right)^{-\theta}$$

$$X_3 \equiv \frac{q_0 z}{X_2(1-\gamma)^{\frac{\epsilon}{\epsilon-1}}\beta N_0^{\beta}(N-1)}$$

We have prove there exist a unique stationarized steady state, (k^*, a^*, p^*) in the complementary case. Now we can linearize our dynamical system around the steady state.

We consider this dynamical system:

$$\begin{aligned}
\dot{p} &= -p\left\{(1-\alpha)(1-\gamma)\phi^{\frac{1}{\epsilon}}\left(\frac{L_2(k,a,p,q_0,N_0)}{K_2(k,a,p,q_0,N_0}\right)^{\beta} + g_P - \rho - \delta\right\} \equiv \mathcal{F}(p,k,a) \\
\dot{k} &= k\left\{\phi\left(\frac{L_2(k,a,p,q_0,N_0)}{K_2(k,a,p,q_0,N_0}\right)^{\alpha}\frac{K_2(k,a,p,q,N_0)}{k} - \delta - g_K - \frac{N_0 p^{-\frac{1}{\theta}}}{k}\right\} \equiv \mathcal{G}(p,k,a) \\
\dot{a} &= a\left\{z\Big(1-u(k,a,p,q_0,N_0)\Big) - g_A - \eta\right\} \equiv \mathcal{H}(p,k,a)
\end{aligned}$$

The linearization around the steady state yields the following Jacobian matrix:

$$\mathcal{J} = \begin{pmatrix} \mathcal{F}_1(p^*,k^*,a^*) & \mathcal{F}_2(p^*,k^*,a^*) & \mathcal{F}_3(p^*,k^*,a^*) \\ \mathcal{G}_1(p^*,k^*,a^*) & \mathcal{G}_2(p^*,k^*,a^*) & \mathcal{G}_3(p^*,k^*,a^*) \\ \mathcal{H}_1(p^*,k^*,a^*) & \mathcal{H}_2(p^*,k^*,a^*) & \mathcal{H}_3(p^*,k^*,a^*) \end{pmatrix}$$

with

$$\begin{aligned}
\mathcal{F}_1(p^*,k^*,a^*) &= 0 \\
\mathcal{F}_2(p^*,k^*,a^*) &= (g_P - \rho - \delta)\beta\tfrac{l^*}{k^*} \\
\mathcal{F}_3(p^*,k^*,a^*) &= 0 \\
\mathcal{G}_1(p^*,k^*,a^*) &= \tfrac{1}{\theta}N_0 p^{*\frac{-1-\theta}{\theta}} \\
\mathcal{G}_2(p^*,k^*,a^*) &= (1-\beta)(1-\gamma)^{\frac{\epsilon}{\epsilon-1}}(\tfrac{l^*}{k^*})^{\beta} - g_K - \delta \\
\mathcal{G}_3(p^*,k^*,a^*) &= 0 \\
\mathcal{H}_1(p^*,k^*,a^*) &= \tfrac{za^*\epsilon u}{p^*} \\
\mathcal{H}_2(p^*,k^*,a^*) &= -\tfrac{za^*u(\gamma-\epsilon)(1-\beta)}{k^*} \\
\mathcal{H}_3(p^*,k^*,a^*) &= \tfrac{zu(\alpha-\epsilon)}{\alpha}
\end{aligned}$$

We compute the trace and the determinant:

$$\begin{aligned}
\mathcal{T} &= \mathcal{G}_2(p^*,k^*,a^*) + \mathcal{H}_3(p^*,k^*,a^*) \\
\mathcal{D} &= -\mathcal{F}_2(p^*,k^*,a^*)\left[\mathcal{G}_1(p^*,k^*,a^*)\mathcal{H}_3(p^*,k^*,a^*)\right]
\end{aligned}$$

Because of the value of $\mathcal{G}_2$ the trace seems to be always positive. However, to have a negative determinant we have to impose some restrictions on α and ϵ. We see clearly that if $\alpha > \epsilon$ the determinant is negative so we have a saddle-path configuration, but if it is not the case we have complete instability of the model with a positive determinant and a positive trace. *QED*

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