Assessing deprivation with ordinal variables: Depth sensitivity and poverty aversion*

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Abstract

The challenges associated with poverty measurement in an axiomatic framework, especially with cardinal variables, have received due attention during the last four decades. However, there is a dearth of literature studying how to meaningfully assess poverty with ordinal variables, capturing the *depth* of deprivations. In this paper, we first axiomatically characterise a class of additively decomposable ordinal poverty measures using a set of basic foundational properties. We then introduce, in a novel effort, a set of properties incorporating different degrees of *poverty aversion* in the ordinal context and characterise relevant subclasses. We further develop related stochastic dominance conditions for all our characterised classes and subclasses of measures. We demonstrate the efficacy of our methods using an empirical illustration studying sanitation deprivation in Bangladesh. Finally, we elucidate how our ordinal measurement framework is related to the burgeoning literature on multidimensional poverty measurement.

Keywords: Ordinal variables, poverty measurement, precedence to poorer people, Hammond transfer, degree of poverty aversion, stochastic dominance.

JEL Codes: I3, I32, D63, O1

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1. Introduction

Around four decades ago, in an influential article titled *Poverty: An Ordinal Approach to Measurement*, Nobel laureate Amartya Sen proposed an axiomatically derived poverty measure to avoid some shortcomings of the traditionally used headcount ratio (Sen, 1976). Sen's approach was ordinal in the sense that his poverty measures assigned an ordinal-rank weight to each poor person's income, an otherwise cardinal variable. Since then, this seminal article has influenced a well-developed literature on poverty measurement involving cardinal variables within an axiomatic framework (Thon, 1979; Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984; Foster and Shorrocks, 1988a,b; Ravallion, 1994; Shorrocks, 1995).

Distances between the values of cardinally measurable variables are meaningful. By contrast, ordinal variables merely consist of ordered categories and cardinal distances between these categories are hard to interpret when numerals are assigned to them, respecting their order or rank.[1] And yet, the practice of using ordinal variables has been on the rise due to the recent surge in interest toward studying deprivation in non-monetary indicators, which are often ordinal in nature (e.g. type of access to basic facilities).[2] However, there may also be instances where ordinal categories of an otherwise cardinally measurable variable may have more policy relevance. For example, it may be of more interest to focus on ordered categories of income, nutritional status, or years of education completed, than the cardinal values of these indicators themselves.

How should poverty be meaningfully assessed with ordinal variables? One straightforward way may be to dichotomise the population into a group of deprived and a group of non-deprived people, and then use the *headcount ratio*. However, this index is widely accused of ignoring the depth of deprivations (Foster and Sen, 1997). For instance, in Sylhet province of Bangladesh between 2007 and 2011, the proportion of population with inadequate sanitation facilities went down from around 70% to nearly 63%; whereas, during the same period, the proportion of people with the worst form of sanitation deprivation ('open defecation') increased significantly, from around 2% to more than 12% (see Table 2 in Section 6).

How can the depth of deprivations be reasonably captured in the case of ordinal variables? The challenges associated with measuring well-being and inequality using an ordinal variable in an axiomatic framework have received due attention during the last few decades (e.g., Mendelson, 1987; Allison and Foster, 2004; Apouey, 2007; Abul Naga and Yalcin, 2008; Zheng, 2011; Kobus and Milos, 2012; Permanyer and D'Ambrosio, 2015; Kobus, 2015; Lazar and Silber, 2013; Yalonetzky, 2013; Gravel et al., 2015). Yet such efforts in the assessment of poverty have been inadequate so far (see, for example, Bennett and Hatzimasoura,

[1]Based on the classification of measurement scales by Stevens (1946), whenever numeral scales are assigned to different ordered categories of an ordinal variable according to the orders or ranks of these categories, any 'order-preserving' or monotonic transformation should leave the scale form invariant. See Roberts (1979) for further in-depth discussions. In this paper, by ordinal variables we simply refer to variables with ordered categories, where numeral scales may not have necessarily been assigned to the categories.

[2]For example, as part of the first Sustainable Development Goals, the United Nations has set the target to not only eradicate *extreme monetary poverty* for all people everywhere, but also to reduce at least by half the proportion of men, women and children of all ages living in *poverty in all its dimensions* by 2030. See http://www.un.org/sustainabledevelopment/poverty/ (accessed in April 2017).

2011; Yalonetzky, 2012).[3] Bennett and Hatzimasoura (2011) characterised a class of ordinal poverty indices, motivated by the seminal Foster et al. (1984) class of indices for cardinal variables. Each ordinal measure in the Bennett-Hatzimasoura class could be expressed as a *weighted* sum of the population proportions in each deprivation category, where the weights are based on normalised ordinal rank shortfalls of the deprivation categories and are *uniquely* determined by a single parameter resembling the poverty aversion parameter of Foster et al. (1984). However, Yalonetzky (2012) showed such weights to be unnecessarily restrictive.

Our paper contributes theoretically to the poverty measurement literature in three ways. First, we axiomatically characterise a class of ordinal poverty measures under a minimal set of well-motivated and desirable properties. Our class consists of measures that are weighted sums of population proportions in deprivation categories and includes the Bennett-Hatzimasoura class as a subclass. Our measures are sensitive to the depth of deprivations, unlike the headcount ratio, are additively decomposable and are bounded between zero and one. These features make them amenable for a broad range of empirical applications.

Second, an adequately designed poverty measure should also ensure that policy makers have additional incentive to provide precedence to those poorer among the poor in the design of poverty alleviation policies so that *the poorest are not left behind*.[4] We operationalise the concept of *precedence to poorer people* with a novel property incorporating a type of *aversion to poverty* in the ordinal context and characterise corresponding sub-classes of ordinal poverty measures. This new property encompasses, as limiting cases, both previous attempts at sensitising ordinal poverty indices to the depth of deprivations (Bennett and Hatzimasoura, 2011; Yalonetzky, 2012) as well as current burgeoning approaches to distributional sensitivity in ordinal frameworks based on Hammond transfers (Hammond, 1976; Gravel et al., 2015).

Third, since each of our classes and subclasses admits a large number of poverty measures, we develop related *stochastic dominance conditions* whose fulfilment guarantees the robustness of poverty comparisons to alternative functional forms and measurement parameters. While a few conditions are the ordinal-variable analogue of existing dominance conditions for cardinal variables (Foster and Shorrocks, 1988b), others are themselves a novel methodological contribution to the literature on stochastic dominance with ordinal variables, to the best of our knowledge.

To demonstrate the efficacy of our approach, we first present an empirical illustration studying the evolution of sanitation deprivation in Bangladesh. Interestingly, our measures are able to discern the instances where the improvements in overall sanitation deprivation did not necessarily include the poorest. We then discuss how our proposed class may be applied in the multidimensional context, where multiple variables are used jointly to assess poverty. We show that many well-known additively decomposable multidimensional poverty indices, based on the counting approach (Townsend, 1979; Atkinson, 2003), have the same aggregation expression as our proposed class of ordinal measures.

[3] We refer to the unidimensional context here. The issue of ordinality has certainly been examined thoroughly in the context of multidimensional poverty measurement (Alkire and Foster, 2011; Bossert et al., 2013; Dhongde et al., 2016; Bosmans et al., 2017). However, even in the multidimensional context, ordinal variables are dichotomised in practice, thereby ignoring the depth of deprivations within indicators.

[4] Poverty measures may affect the incentives of policy makers during poverty alleviation (Zheng, 1997).

The rest of the paper proceeds as follows. We present the notation and basic measurement framework in Section 2. In Section 3, we introduce the key desirable properties and characterise the class of ordinal poverty measures. Section 4 introduces the concept of precedence to poorer people and characterises the subclass of relevant poverty indices. Section 5, then, develops stochastic dominance conditions for the characterized class and subclasses of poverty measures. Section 6 provides an empirical illustration analysing sanitation deprivation in Bangladesh. Section 7 elucidates how our ordinal measurement framework can contribute to the burgeoning literature of multidimensional poverty measurement. Section 8 concludes.

2. Notation and framework

Suppose there is a *social planner* whose objective is to assess poverty in a hypothetical society consisting of $N \in \mathbb{N}$ individuals, where $\mathbb{N}$ is the set of positive integers. We denote the level of well-being of person n by $x_n \in \mathbb{R}_+$ for all $n = 1, \ldots, N$, where $\mathbb{R}_+$ is the set of non-negative real numbers. We denote the vector of individual well-being levels by $\mathbf{x} = (x_1, \ldots, x_N)$, the set of all individual well-being vectors of population size N by $\mathbf{X}_N$, and the set of all individual well-being vectors of any population size by $\mathbf{X}$. The set of all individuals in $\mathbf{x}$ is denoted by $\mathbf{N}(\mathbf{x})$.

The actual level of individual well-being in this hypothetical society may often be unmeasurable or somehow unobservable to the social planner. Instead, the social planner may merely observe a set of ordered categories. For instance, self-reported health status may only include response categories, such as 'good health', 'fair health', 'poor health', and 'very poor health'. Similarly, there are also instances where the ordinal categories of an otherwise cardinal variable, such as - the *Body Mass Index* (BMI) for assessing nutritional status or the *years of schooling completed* for assessing the level of educational attainment, has more policy relevance. Even though the BMI is cardinal, the differences between its cardinal values may not have the same interpretation. According to the World Health Organisation (WHO), both the BMIs of 15.4 and 15.9 mean 'severe thinness', but a BMI of 18.4 means 'mild thinness' and a BMI of 18.9 means 'normal weight', despite the same cardinal differences. Notably, a 'severely thin' person is less well nourished than a 'moderately thin' person and both are less well nourished than a 'normal weight' person.

Suppose, there is a fixed set of $S \in \mathbb{N}\backslash\{1\}$ ordered categories $c_1, \ldots, c_S$, such that $c_{s-1} \succ_D c_s$ for all $s = 2, \ldots, S$, where $\succ_D$ is a binary, transitive and reflexive relation whereby category c_{s-1} represents a worse-off situation (or higher deprivation) than category c_s. Thus, c_S is the category reflecting highest well-being (or least deprivation) and c_1 is the state reflecting lowest well-being or highest deprivation. Suppose, for example, a society's well-being is assessed by the education dimension and the observed ordered categories are: No education, Primary education, Secondary education, and Higher education, such that No education $\succ_D$ Primary education $\succ_D$ Secondary education $\succ_D$ Higher education. Then, c_4 = Higher education and c_1 = No education. We denote the set of all S categories by $\mathbf{C} = \{c_1, c_2, \ldots, c_S\}$ and the set of all categories excluding the category of least deprivation c_S by $\mathbf{C}_{-S} = \mathbf{C} \setminus \{c_S\}$.

We denote the set of individuals in $\mathbf{x}$ experiencing category c_s by $\Omega_s(\mathbf{x})$, such that $\Omega_s(\mathbf{x}) \cap$

$\Omega_{s'}(\mathbf{x}) = \varnothing$ for all $s \neq s'$ and $\cup_{s=1}^{S}\Omega_s(\mathbf{x}) = \mathbf{N}(\mathbf{x})$. Let $p_s(\mathbf{x})$ denote the proportion of the population in $\mathbf{x}$ experiencing category c_s; that is, the proportion of overall population in $\Omega_s(\mathbf{x})$. Then, by definition, $p_s(\mathbf{x}) \geq 0$ for all s and $\sum_{s=1}^{S} p_s(\mathbf{x}) = 1$. We denote the proportions of population in $\mathbf{x}$ across S categories by $\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_S(\mathbf{x}))$.

It is customary in poverty measurement to define a poverty threshold identifying the poor and the non-poor populations (Sen, 1976). Suppose, the social planner decides that category c_k for any $1 \leq k < S$ and $k \in \mathbb{N}$ be the *poverty threshold*, so that people experiencing categories $c_1, \ldots, c_k$ are identified as *poor*; whereas people experiencing categories $c_{k+1}, \ldots, c_S$ are identified as *non-poor*. We assume that at least one category reflects absence of poverty, as this restriction is both intrinsically reasonable and required for stating certain properties in Section 3. When $k = 1$, only category c_1 reflects poverty and for any $\mathbf{x} \in \mathbf{X}$, in this case, $p_1(\mathbf{x})$ is the proportion of population identified as poor. For every $\mathbf{x} \in \mathbf{X}$ and for every $c_k \in \mathbf{C}_{-S}$, we denote the *set of poor population* by $\mathbf{Z}^P(\mathbf{x}; c_k) = \cup_{s=1}^{k}\Omega_s(\mathbf{x})$, the *set of non-poor population* by $\mathbf{Z}^{NP}(\mathbf{x}; c_k) = \cup_{s=k+1}^{S}\Omega_s(\mathbf{x})$, and the *proportion of poor population* or the *headcount ratio* by $H(\mathbf{x}; c_k) = \sum_{s=1}^{k} p_s(\mathbf{x})$.

A *poverty measure* $P(\mathbf{x}; c_k)$ is defined as $P : \mathbf{X} \times \mathbf{C}_{-S} \to \mathbb{R}_+$. In words, a poverty measure is a mapping from the set of individual well-being vectors and the set of poverty thresholds to the real line.

We introduce some additional concepts and notation that will be useful when stating some of the properties in the next section. First, for any $j \in \mathbb{N}\backslash\{1\}$, a *permutation matrix* $\mathcal{P}^j$ is a $j \times j$ non-negative square matrix with only one element in each row and each column being equal to one and the rest of the elements being equal to zero. Then, for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^j$, we say that $\mathbf{b}$ is obtained from $\mathbf{a}$ by *permutation* if $\mathbf{b} = \mathbf{a}\mathcal{P}^j$, where a permutation simply changes the position of the elements within a vector. Second, for any $\mathbf{a} \in \mathbb{R}^j$ and for any $\mathbf{b} \in \mathbb{R}^{r \times j}$, where $r \in \mathbb{N}\backslash\{1\}$ and $j \in \mathbb{N}$, we say that $\mathbf{b}$ is obtained from $\mathbf{a}$ by *replication* whenever $\mathbf{b} = (\mathbf{a}, \ldots, \mathbf{a})$. Note that a replication simply creates a multiplication of every element in a vector by $r > 1$ times to obtain another vector.

In this paper, we will also be interested in exploring the relationship between the overall poverty evaluation and the subgroup poverty evaluation, which requires some subgroup notation. Suppose, the entire society with individual well-being vector $\mathbf{x} \in \mathbf{X}_N$ is partitioned into $M \in \mathbb{N}\backslash\{1\}$ mutually exclusive and collectively exhaustive population subgroups, such that $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^M)$. The individual well-being vector of subgroup m is denoted by $x^m \in \mathbf{X}_{N^m}$ for all m, where the population size of subgroup m is denoted by $N^m \in \mathbb{N}$, such that $\sum_{m=1}^{M} N^m = N$.

3. Properties and axiomatic characterisation

Can the information available for the ordered categories of an ordinal variable be meaningfully aggregated to obtain a *cardinal poverty measure*? The answer to this question depends on certain desirable properties that the poverty measure P could be demanded to satisfy. We introduce eight properties in this section. The first four properties are stated in a way that is customary in the literature, defined on the actual levels of individual well-being. The rest of the four properties are adapted to the ordinal setting, aligned with the assumption that the social planner only observes the ordered categories instead of the actual levels of well-being. We then derive and axiomatically characterise a class of ordinal poverty measures in Theorem 3.1, using the result presented in Lemma 3.1.

The first property is *continuity*, which requires a poverty measure to be jointly continuous on individual well-being levels and poverty thresholds. In other words, a poverty measure should change continuously owing to a change in a poor individual's well-being level:

Continuity (CON) P is jointly continuous on $\mathbf{X}$ and $\mathbf{C}_{-S}$.

The second property is *anonymity*, which requires each person's identity to remain anonymous for the purpose of poverty measurement. Thus, merely shuffling the individual well-being levels of people within a society, keeping population size unchanged, should not alter the social poverty evaluation:

Anonymity (ANO) For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y} = \mathbf{x}\mathcal{P}^N$ then $P(\mathbf{x}; c_k) = P(\mathbf{y}; c_k)$.

The third property is *population principle*, which requires that a mere duplication of individual well-being levels should not alter the poverty evaluation. This property allows us to compare societies with different population sizes:

Population Principle (POP) For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y}$ is obtained from $\mathbf{x}$ by *replication*, then $P(\mathbf{x}; c_k) = P(\mathbf{y}; c_k)$.

The fourth property, *subgroup consistency*, is due to Foster and Shorrocks (1991). This property requires that if any individual well-being vector is partitioned into two or more mutually exclusive and collectively exhaustive population subgroups, and if poverty increases strictly in any one population subgroup while poverty does not decrease in any other population subgroup(s), then overall poverty must increase. It is a policy-relevant property which prevents inconsistent poverty evaluations:

Subgroup Consistency (SCN) For any $M \in \mathbb{N}/\{1\}$, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$ such that $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^M)$ and $\mathbf{y} = (\mathbf{y}^1, \ldots, \mathbf{y}^M)$ where $\mathbf{x}^m, \mathbf{y}^m \in \mathbf{X}_{N^m}$ for some $N^m \in \mathbb{N}$ for all $m = 1, \ldots, M$, and for any $c_k \in \mathbf{C}_{-S}$, if $P(\mathbf{y}^j; c_k) > P(\mathbf{x}^j; c_k)$ but $P(\mathbf{y}^m; c_k) \geq P(\mathbf{x}^m; c_k)$ for all $m \neq j$, then $P(\mathbf{y}; c_k) > P(\mathbf{x}; c_k)$.

The first four properties allow a poverty measure $P : \mathbf{X} \times \mathbf{C}_{-S} \to \mathbb{R}$ to be presented in additively separable form as in Equation 1 within Lemma 3.1. We refer to the class of poverty measures presented in Equation 1 by $\bar{\mathcal{P}}$.

Lemma 3.1 A poverty measure $P : \mathbf{X} \times \mathbf{C}_{-S} \to \mathbb{R}$ satisfies properties CNT, ANO, POP, and SCN if and only if

$$P(\mathbf{x}; c_k) = F\left[\frac{1}{N}\sum_{n=1}^{N}\phi(x_n)\right]; \tag{1}$$

where F is continuous and increasing and ϕ is continuous.

Proof. The proof follows directly from Foster and Shorrocks (1991). ■

The remaining four properties are adapted to respect the ordinal framework. The fifth property, *ordinal categorisation*, requires that if all individuals in two different societies with equal population sizes experience the same category, then the two societies should experience the same level of poverty. What is the justification for this assumption? Note that the social planner may not observe or be interested in the actual levels of individuals' well-being; so if the social planner observes all individuals to experience the same category across two different societies, then she assumes the levels of poverty across these societies to be the same:

Ordinal Categorisation (ORC) For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $n \in \Omega_s(\mathbf{x})$ for all $n \in \mathbf{N}(\mathbf{x})$ and $n \in \Omega_s(\mathbf{y})$ for all $n \in \mathbf{N}(\mathbf{y})$ and for some $s \in \{1, \ldots, S\}$, then $P(\mathbf{x}; c_k) = P(\mathbf{y}; c_k)$.

The sixth property is *ordinal monotonicity*, which requires that if the well-being level of a poor person improves so that the person experiences a better category of well-being, then poverty should be lower. The formal statement of the property requires that if a poor person moves from a category c_s reflecting poverty to a less deprived state $c_{s'}$, while the well-being levels of every other person remain unchanged, then poverty should fall:

Ordinal Monotonicity (OMN) For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{s'}(\mathbf{y})$ for any $s' > s$, while $x_n = y_n$ for all $n \neq n'$, then $P(\mathbf{y}; c_k) < P(\mathbf{x}; c_k)$.

The seventh property is *convergence to headcount ratio.* The property requires that whenever there is only one state reflecting poverty (i.e. c_1), then the poverty measure should be equal to the headcount ratio $H(\cdot; c_1) = p_1(\cdot)$. In other words, we assume that whenever there is only one category reflecting poverty and the rest reflecting absence of poverty, then the headcount ratio becomes a sufficient statistic for the assessment of poverty. In fact, in this situation, any functional transformation of the headcount ratio would not add any meaningful information to the poverty assessment while being inferior in terms of intuitive interpretation:

Convergence to Headcount Ratio (CHR) For any $\mathbf{x} \in \mathbf{X}$ and $c_1 \in \mathbf{C}_{-S}$,
$P(\mathbf{x}; c_1) = p_1(\mathbf{x})$.

The final property, *focus*, is essential for a poverty measure. It requires that, *ceteris paribus*, change in the individual well-being levels among the non-poor should not alter poverty evaluations as long as non-poor people remain in that status. Given that the social planner cannot observe the actual well-being levels of individuals, it is required that as long as the set of poor people remains unchanged within each of the k categories reflecting poverty, the level of poverty should be the same. Note that the set of non-poor people may remain unchanged or may be different across the $S-k$ categories *not* reflecting poverty, but this should not matter for poverty evaluation:

Focus (FOC) For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$ and for any $c_k \in \mathbf{C}_{-S}$, if $\Omega_s(\mathbf{x}) = \Omega_s(\mathbf{y})$ $\forall s \leq k$, then $P(\mathbf{x}; c_k) = P(\mathbf{y}; c_k)$.

These final four properties then lead to the class $\mathcal{P}$ of poverty measures as in Theorem 3.1 using the result presented in Lemma 3.1:

Theorem 3.1 A poverty measure $P \in \bar{\mathcal{P}}$ satisfies properties OMN, ORC, CHR and FOC if and only if

$$P(\mathbf{x}; c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s \tag{2}$$

where $\omega_1 = 1$, $\omega_{s-1} > \omega_s > 0$ for all $s = 2, ..., k$ whenever $k \geq 2$, and $\omega_s = 0$ for all $s > k$.

Proof. See Appendix A1. ■

Theorem 3.1 is quite powerful in the sense that the only class of additively separable poverty measures in $\bar{\mathcal{P}}$ that satisfy the set of four ordinal properties is the weighted sum of the population proportions in $\mathbf{p}(\mathbf{x})$, where the weights are *non-negative* for all categories, *strictly positive* for the deprived categories, and *unity* for the most deprived category. We refer to weights ω_s's as *ordering weights* and to $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$ as the *ordering weighting vector*.

In order to satisfy the key OMN property, the ordering weights increase with deprived categories representing higher levels of deprivation. In practice, the ordering weights may take various forms. One example may be drawn from Bennett and Hatzimasoura (2011), where each deprivation category is assigned ordering weights based on the relative deprivation ranks. In this case, category s is assigned an ordering weight equal to $[(k-s+1)/k]^\theta$ for all $s = 1, \ldots, k$ and for some $\theta > 0$. Thus, the least deprived category c_k receives an ordering weight of $\omega_k = 1/k^\theta$; whereas, the most deprived category c_1 receives an ordering weight of $\omega_1 = 1$. Whichever forms the ordering weights take, nonetheless, Theorem 3.1 clearly requires that the most deprived category must be assigned an ordering weight equal to one and any category not representing poverty must be assigned a 'null ordering weight'.

The class of poverty measures in Equation 2 bears certain policy-relevant features. First, the poverty measure is *additively decomposable*, which means that the overall poverty measure of the society may be expressed as a population-weighted average of the population subgroup poverty measures. Note that this property is not an axiomatic assumption, but stems logically from the foundational properties. Second, the poverty measure is conveniently *normalised* between zero and one. The poverty measure is equal to zero only in a society where nobody is poor; whereas, the poverty measure is equal to one only whenever everybody in the society experiences the worst possible deprivation category c_1. Again, note that this normalisation behaviour is not an axiomatic assumption, but a logical conclusion from the foundational properties (by contrast to Bennett and Hatzimasoura (2011), who proposed weights based on normalised ranks before characterising their class of measures). Third, the poverty measure efficiently boils down to the *headcount ratio* either when the poverty threshold is represented by the most deprived category or whenever the underlying ordinal variable has merely two categories.

4. Providing precedence to the poorer among the poor

Although the poverty measures in Equation (2) satisfy certain desirable properties and bear some policy relevant features, they do not ensure that the poorest among the poor population receive precedence over the less poor population in poverty alleviation efforts. Providing precedence to those poorer has long been considered in the literature to be equivalent to the *egalitarian view* of requiring poverty measures to be sensitive to redistribution of achievements among the poor (Sen, 1976; Foster et al., 1984; Zheng, 1997). This view is tantamount to stating that a poverty measure should decrease whenever inequality among the poor is reduced through rank-preserving *Pigou-Dalton transfers*. However, this view has been questioned by Esposito and Lambert (2011), who argue that the concept of providing precedence to poorer people is actually more aligned with the *prioritarian view*, which states that "benefiting people matters more the worse off these people are" (Parfit, 1997, p. 213). How different are egalitarian and prioritarian views from each other? Fleurbaey (2015, p. 208) argues that these two views fundamentally lead to similar conclusions and "a prioritarian will always find some egalitarians on her side".[5]

We introduce certain properties that reflect different ways of providing precedence to poorer people in the ordinal framework.[6] The first property, which we refer to as *weakest precedence to poorer people*, requires that, *ceteris paribus*, moving a poorer person to an adjacent less deprived category leads to a larger reduction in poverty than moving a less poor person to a respectively adjacent less deprived category. Conceptually, this property is analogous to the Pigou-Dalton transfer principle in an ordinal setting:

Weakest Precedence to Poorer People (PRE-W) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, and for any $c_k \in \mathbf{C}_{-S}$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{s+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n'' \in \Omega_{t+1}(\mathbf{z})$ for some $t > s$ and $n'' \neq n'$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; c_k) < P(\mathbf{z}; c_k)$.

[5]For an application of the prioritarian concept to the multidimensional context, see Bosmans et al. (2017).

[6]We have defined only the strict versions of these properties, requiring poverty to be strictly lower in the aftermath of specific pro-poorest distributional change. Consequently the ensuing results impose strict inequality restrictions on weights. However, these strict restrictions may be relaxed with alternative versions if the latter only require poverty to be simply not higher due to the same pro-poorest distributional change.

The PRE-W property presents a minimal criterion for providing precedence to poorer people. What if a policy maker faces the possibility of improving the well-being of a poorer person by one category vis-à-vis improving the well-being of a richer person by several categories? To ensure that the policy maker still chooses to improve the situation of the poorer person in these cases, we introduce the property of *strongest precedence to poorer people*. This property requires that, *ceteris paribus*, moving a poorer person to a less deprived category leads to a larger reduction in poverty than moving a less poor person to a less deprived category. Note here that the improvement is not restricted to a particular number of adjacent categories.

Strongest Precedence to Poorer People (PRE-S) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, for any $c_k \in \mathbf{C}_{-S}$, and for any $s < s' \leq t < t'$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{s'}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n'' \in \Omega_{t'}(\mathbf{z})$ and $n'' \neq n'$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; c_k) < P(\mathbf{z}; c_k)$.

Conceptually, the PRE-S property is analogous to the notion of 'Hammond transfer' (Hammond, 1976; Gravel et al., 2015), which essentially involves, simultaneously, an improvement of a poor person's situation and a deterioration of a less poor person's situation, such that their well-being ranks are not reversed. Importantly, unlike Pigou-Dalton transfers and thus unlike the PRE-W property, the number of categories between s and s' does not need to be the same as the number of categories between t and t' in the case of PRE-S. An ordinal poverty measure satisfying property PRE-S also satisfies property PRE-W, but the reverse is not true. A policy maker supporting property PRE-S over property PRE-W should be considered more poverty averse.

We can actually generalise this framework (i.e. PRE-S and PRE-W) to incorporate a *degree of precedence* to poorer people.[7] This general property PRE-α, referred to as *precedence to poorer people of order* α, requires that, *ceteris paribus*, moving a poorer person to an adjacent less deprived category leads to a larger reduction in poverty than moving a less poor person up to an $\alpha\,(\geq 1)$ number of adjacent less deprived categories. For example, consider a situation where $S = 7$, $k = 6$ and $\alpha = 3$. Poverty falls faster according to poverty measures satisfying PRE-3 if a poorer person moves from category c_1 to category c_2 than if a less poorer person moves from, say, category c_2 to any of the three categories c_3, c_4, or c_5. It should be noted, however, that it may not always be feasible to improve up to three categories, e.g., moving from category c_5 to category c_8 in this example.

Precedence to Poorer People of Order $\boldsymbol{\alpha}$ (PRE-$\boldsymbol{\alpha}$) For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $k \geq 2$, for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, and for any $c_k \in \mathbf{C}_{-S}$, if (i) $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{s+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$, and (ii) $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n'' \in \Omega_{t'}(\mathbf{z})$ for some $t > s$, $t' = \min\{t + \alpha, S\}$ and $n'' \neq n'$, while $x_n = z_n$ for all $n \neq n''$, then $P(\mathbf{y}; c_k) < P(\mathbf{z}; c_k)$.

[7] The concept is analogous to the degree of poverty or inequality aversion in the cardinal poverty measurement literature (Clark et al., 1981; Chakravarty, 1983; Foster et al., 1984), but not technically identical.

Note that PRE-1 is essentially the PRE-W property. This is the case where the social planner is least poverty averse. As the value of α increases, the social planner's poverty aversion also rises. In this framework, the social planner's poverty aversion is highest at $\alpha = k-1$. We shall show that PRE-α for $\alpha = k-1$ leads to the same sub-class of ordinal poverty measures as the PRE-S property. The PRE-α property imposes further restrictions on the class of measures in Theorem 3.1. In Theorem 4.1, we present the sub-class of measures $\mathcal{P}_\alpha$ that satisfy the PRE-α property:

Theorem 4.1 For any $k \geq 2$ and for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-α if and only if:

a. $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ $\forall s = 2, \ldots, k-\alpha$ and $\omega_{s-1} > 2\omega_s$ $\forall s = k-\alpha+1, \ldots, k$ whenever $\alpha \leq k-2$.

b. $\omega_{s-1} > 2\omega_s$ $\forall s = 2, \ldots, k$ whenever $\alpha = k-1$.

Proof. See Appendix A2. ■

Theorem 4.1 presents various subclasses of indices based on the degree of poverty aversion α, which we denote as $\mathcal{P}_\alpha$. In order to provide precedence to poorer people, the ordering weights must be convex. Corollary 4.1 presents the special case of $\mathcal{P}_1$, featuring the least poverty averse social planner:

Corollary 4.1 For any $k \geq 2$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-W (i.e. PRE-1) if and only if $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 2, ..., k-1$ and $\omega_{k-1} > 2\omega_k$.

Proof. The result follows directly from Theorem 4.1 by setting $\alpha = 1$. ■

To provide precedence to poorer people in the spirit of property PRE-W, the ordering weights must be least convex, such that the difference $\omega_{s-1} - \omega_s$ is larger than the subsequent difference $\omega_s - \omega_{s+1}$, in addition to the restrictions imposed by Theorem 3.1. Suppose, we summarise the ordering weights by: $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_S)$. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega} = (1, 0.8, 0.5, 0, 0)$ and $\boldsymbol{\omega}' = (1, 0.5, 0.2, 0, 0)$, where $k = 3$. The ordering weights in $\boldsymbol{\omega}$ fulfill all properties presented in Theorem 3.1, but the largest reduction in poverty is obtained whenever a poor person moves from the least poor category to the adjacent non-poor category. By contrast, ordering weights in $\boldsymbol{\omega}'$ require that the largest reduction in poverty be obtained whenever a poor person moves from the poorest category to the adjacent second poorest category. Thus, unlike the ordering weights in $\boldsymbol{\omega}$, the ordering weights in $\boldsymbol{\omega}'$ make sure that poorer people receive precedence.

Next we present the subclass of poverty measures that satisfy property PRE-S, i.e. $\mathcal{P}_S$:

Theorem 4.2 For any $k \geq 2$, a poverty measure $P \in \mathcal{P}$ additionally satisfies property PRE-S if and only if $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$.

Proof. See Appendix A3. ■

The additional restriction on the ordering weights in Theorem 4.2 effectively prioritises the improvement in a poorer person's situation over improvement of any extent in a less poor person's situation. Let us consider an example involving five categories and two ordering weight vectors: $\boldsymbol{\omega}^1 = (1, 0.6, 0.3, 0.1, 0)$ and $\boldsymbol{\omega}^2 = (1, 0.48, 0.23, 0.1, 0)$, where $k = 4$. Clearly, both sets of weights in $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ satisfy the restriction in Corollary 4.1 that $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+1}$ for all $s = 1, \ldots, k$. However, the ordering weights in $\boldsymbol{\omega}^1$ do not satisfy the restriction in Theorem 4.2, since $\omega_1^1 < 2\omega_2^1$; whereas the ordering weights in $\boldsymbol{\omega}^2$ do satisfy the restriction in Theorem 4.2 as $\omega_{s-1} > 2\omega_s$ for all $s = 1, \ldots, k$.

Here it is worth pointing out that, remarkably, the subclasses $\mathcal{P}_S$ (Theorem 4.2) and $\mathcal{P}_{k-1}$ (Theorem 4.1 when $\alpha = k-1$) are identical; even though the distributional changes involved in the PRE-α property are only specific cases of the broader Hammond transfers involved in axiom PRE-S. Besides being of interest in itself, this perfect overlap between the subclasses of indices will prove useful in the next section because by deriving the dominance conditions for the subclasses $\mathcal{P}_\alpha$, we will also obtain the relevant dominance conditions for subclass $\mathcal{P}_S$.

5. Dominance conditions

In the previous two sections, we introduced the class of poverty measures $\mathcal{P}$ and its subclasses $\mathcal{P}_\alpha$ and $\mathcal{P}_S$. The main parameters for these measures are the set of ordering weights $\{\omega_1, \ldots, \omega_k\}$, the poverty threshold category c_k, and the poverty aversion parameter α. It is thus natural to inquiry into the circumstances under which ordinal poverty comparisons are robust to the alternative ordering weights as well as to the alternative poverty threshold categories. In this section, first we introduce the first-order dominance conditions relevant to $\mathcal{P}$, followed by the second-order dominance conditions for $\mathcal{P}_\alpha$ for all α.

In order to state the conditions we introduce some additional notation. First, we define the cumulative distribution function (CDF) of any distribution $\mathbf{x} \in \mathbf{X}$ as $F(\mathbf{x}; c_s) \equiv \sum_{\ell=1}^{s} p_\ell(\mathbf{x})$ for all $s = 1, \ldots, S$. Clearly, $F(\mathbf{x}; c_1) = p_1(\mathbf{x})$ and $F(\mathbf{x}; c_S) = 1$. We denote the *difference operator* by Δ, and for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, express $\Delta P(\mathbf{x}, \mathbf{y}; c_k) \equiv P(\mathbf{x}; c_k) - P(\mathbf{y}; c_k)$ where $P(\cdot; c_k) = \sum_{s=1}^{S} p_s(\cdot)\omega_s$ from Equation (2); $\Delta F(\mathbf{x}, \mathbf{y}; c_s) \equiv F(\mathbf{x}; c_s) - F(\mathbf{y}; c_s)$; and $\Delta p_s(\mathbf{x}, \mathbf{y}) \equiv p_s(\mathbf{x}) - p_s(\mathbf{y})$. For notational convenience, we will often refer to $\Delta P(\mathbf{x}, \mathbf{y}; c_k)$ as ΔP_k, $\Delta F(\mathbf{x}, \mathbf{y}; c_s)$ as ΔF_s, and $\Delta p_s(\mathbf{x}, \mathbf{y})$ as Δp_s.

5.1. First-order dominance conditions

Theorem 5.1 provides the first-order dominance conditions relevant to all measures in class $\mathcal{P}$ for a *given* poverty threshold category $c_k \in \mathbf{C_{-S}}$:

Theorem 5.1 For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for any $k \geq 1$, $\Delta P(\mathbf{x}, \mathbf{y}; c_k) < 0$ for all $P \in \mathcal{P}$ for a given $c_k \in \mathbf{C_{-S}}$ if and only if $\Delta F(\mathbf{x}, \mathbf{y}; c_s) \leq 0$ for all $s \leq k$ with at least one strict inequality.

Proof. See Appendix A4. ■

Theorem 5.1 states that poverty in one distribution $\mathbf{x} \in \mathbf{X}$ is strictly lower than in another distribution $\mathbf{y} \in \mathbf{X}$ for a chosen poverty threshold category $c_k \in \mathbf{C_{-S}}$ and for all measures $P \in \mathcal{P}$ if and only if the CDF of $\mathbf{x}$ is nowhere above and at least once below the CDF of $\mathbf{y}$ up to category c_k. In other words, the poverty comparison for a particular poverty threshold category c_k is robust to all poverty measures $P \in \mathcal{P}$ if and only if $H(\mathbf{x}; c_s) \leq H(\mathbf{y}; c_s)$ for all $s \leq k$ and $H(\mathbf{x}; c_s) < H(\mathbf{y}; c_s)$ for at least one $s \leq k$.

Corollary 5.1 provides the first-order dominance condition relevant to any measure $P \in \mathcal{P}$ for *all* $c_k \in \mathbf{C_{-S}}$:

Corollary 5.1 For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for any $k \geq 1$, $\Delta P(\mathbf{x}, \mathbf{y}; c_k) < 0$ for any $P \in \mathcal{P}$ and for all $c_k \in \mathbf{C_{-S}}$ if and only if $\Delta F(\mathbf{x}, \mathbf{y}; c_s) \leq 0$ for all $s = 2, \ldots, k$ and $\Delta p_1(\mathbf{x}, \mathbf{y}) < 0$.

Proof. The sufficiency part is straightforward and follows from Equation A12. We prove the necessary condition as follows. First, consider $k = 1$. Then, $\Delta P_1 < 0$ only if $\Delta F_1 < 0$ or, equivalently, $\Delta p_1 < 0$. Subsequently, the requirement that $\Delta F(\mathbf{x}, \mathbf{y}; c_s) \leq 0$ for every $s = 2, \ldots, k$ follows from Theorem 5.1. ■

Interestingly, poverty in distribution $\mathbf{x}$ is lower than poverty in distribution $\mathbf{y}$ for any $P \in \mathcal{P}$ and for all possible poverty threshold categories if and only if $H(\mathbf{x}; c_s) \leq H(\mathbf{y}; c_s)$ for all $s \leq k$ and $H(\mathbf{x}; c_1) < H(\mathbf{y}; c_1)$. The results in Theorem 5.1 and Corollary 5.1 are the ordinal versions of the headcount-ratio orderings for continuous variables derived by Foster and Shorrocks (1988b).

5.2. Second-order dominance conditions

In this section, in Theorem 5.2 we first present the second-order general dominance conditions relevant to any measure in subclass $\mathcal{P}_\alpha$ for $\alpha \geq 1$ for a *given* poverty threshold category $c_k \in \mathbf{C_{-S}}$ such that $k \geq 2$:

Theorem 5.2 For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, for any $k \geq 2$, and for any $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k-1$, $\Delta P(\mathbf{x}, \mathbf{y}; c_k) < 0$ for all $P \in \mathcal{P}_\alpha$ for a given $c_k \in \mathbf{C_{-S}}$:

a. **if** $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0$ for all $s = 1, \ldots, k$ with at least one strict inequality; and

b. **only if**:

   i. $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0$ for all $s = 1, \ldots, k$ with at least one strict inequality, when $\alpha = 1$.

   ii. $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0$ for all $s = 1, \ldots, k - \alpha + 1$ and $(\sum_{\ell=1}^{k-\alpha} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell)) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell)) + 2^{1-\alpha} \Delta F(\mathbf{x}, \mathbf{y}; c_k) \leq 0$ with at least one strict inequality; when $2 \leq \alpha \leq k-1$.

Proof. See Appendix A5. ■

Thus, poverty in $\mathbf{x}$ is lower than poverty in $\mathbf{y}$ for a given poverty threshold category, for all $P \in \mathcal{P}_\alpha$, and for a given α if all cumulations of the cumulative distribution of $\mathbf{x}$ is nowhere above and at least once strictly below that of $\mathbf{y}$ up to the poverty threshold category c_k. Note that this second-order sufficiency requirement is the same no matter the value of α. However, the sufficient condition is not necessary. The necessary condition depends on the value of α. It is only for $\alpha = 1$, i.e. the case of $P \in \mathcal{P}_1$ satisfying property PRE-W, that the necessary condition is identical to the sufficient condition. In fact, the condition for $\alpha = 1$ is the ordinal version of the 'P_2' poverty ordering due to Foster and Shorrocks (1988b).

Unlike the case of $\alpha = 1$, the necessary conditions diverge from the sufficient conditions whenever $\alpha \geq 2$. The number of restrictions to check for necessity decreases as the value of α increases. For $\alpha = k-1$, only three such restrictions must be checked. Corollary 5.2 presents the necessary condition for $\Delta P_k < 0$ when $\alpha = k-1$ or whenever the poverty measures satisfy property PRE-S:

Corollary 5.2 For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and for any $k \geq 2$, $\Delta P(\mathbf{x}, \mathbf{y}; c_k) < 0$ for all $P \in \mathcal{P}_S$ for a given $c_k \in \mathbf{C_{-S}}$ only if $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0$ for $s = 1, 2$ and $\Delta F(\mathbf{x}, \mathbf{y}; c_1) + (\sum_{\ell=2}^{k-1} 2^{1-\ell} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell)) + 2^{2-k} \Delta F(\mathbf{x}, \mathbf{y}; c_k) \leq 0$ with at least one strict inequality.

Proof. It is straightforward to verify from Theorem 5.2 by setting $\alpha = k-1$. ■

Finally, Corollary 5.3 provides the second-order dominance conditions relevant to any measure $P \in \mathcal{P}_\alpha$ for *all* $c_k \in \mathbf{C_{-S}}$ such that $k \geq 2$:

Corollary 5.3 For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\Delta P(\mathbf{x}, \mathbf{y}; c_k) < 0$ for any $P \in \mathcal{P}_\alpha$ and for all $c_k \in \mathbf{C_{-S}} \backslash \{c_1\}$

a. **if** *either* $(\sum_{\ell=1}^{2} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0 \cap \Delta p_1(\mathbf{x}, \mathbf{y}) < 0)$ *or* $(\sum_{\ell=1}^{2} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) < 0 \cap \Delta p_1(\mathbf{x}, \mathbf{y}) \leq 0)$ whenever $k = 2$ and additionally $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0 \; \forall s = 3, \ldots, k$ whenever $k \geq 3$; and

b. **only if** either $(\sum_{\ell=1}^{2} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0 \cap \Delta p_1(\mathbf{x}, \mathbf{y}) < 0)$ or $(\sum_{\ell=1}^{2} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) < 0 \cap \Delta p_1(\mathbf{x}, \mathbf{y}) \leq 0)$ whenever $k = 2$ and additionally $\sum_{\ell=1}^{s} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell) \leq 0$ for all $s = 3, \ldots, k-\alpha+1$ and $(\sum_{\ell=1}^{k-\alpha} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell)) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F(\mathbf{x}, \mathbf{y}; c_\ell)) + 2^{1-\alpha} \Delta F(\mathbf{x}, \mathbf{y}; c_k) \leq 0$ whenever $k \geq 3$.

Proof. First consider the case when $k = 2$. From Equation A13, we obtain $\Delta P_2 = (\omega_1 - 2\omega_2)\Delta F_1 + \omega_2(\Delta F_1 + \Delta F_2)$. It is easy to verify that in order to have $\Delta P_2 < 0$, it is both necessary and sufficient that either $(\sum_{\ell=1}^{2} \Delta F_\ell \leq 0 \cap \Delta p_1 < 0)$ or $(\sum_{\ell=1}^{2} \Delta F_\ell < 0 \cap \Delta p_1 \leq 0)$. The additional sufficient and necessary condition for $k \geq 3$ follows from Theorem 5.2. ■

In summary, the robustness of poverty comparisons for various classes and sub-classes of ordinal measures introduced in Sections 3 and 4 can be assessed with a battery of dominance tests based on the theorems and corollaries presented in this section.

6. Empirical illustration: Sanitation deprivation in Bangladesh

We now present an empirical illustration in order to showcase the efficacy of our proposed measurement method. In the current global development context, both the United Nations through the Sustainable Development Goals[8] and the World Bank through their Report of the Commission on Global Poverty (World Bank, 2017) have acknowledged the need for assessing, monitoring, and alleviating poverty in multiple dimensions besides the monetary dimension. In practice, most non-income dimensions are assessed by ordinal variables. In this section, we show how our measurement tools may be applied to analyse inter-temporal sanitation deprivation in Bangladesh.

For our analysis, we use the nationally representative Demographic Health Survey (DHS) datasets of Bangladesh for years 2007, 2011 and 2014. While computing the estimates and the standard errors, we incorporate the sampling weights as well as respect the survey design.[9] Excluding the non-usual residents, we were able to use the information on 50,215 individuals from 10,398 households in the 2007 survey, 79,483 individuals from 17,139 households in the 2011 survey, and 77,680 individuals from 17,299 households in the 2014 survey.

Table 1: The five ordered categories of access to sanitation facilities

Category	Description
Open defecation	Human faeces disposed of in fields, forests, bushes, open bodies of water, beaches or other open spaces or disposed of with solid waste
Unimproved	Pit latrines without a slab or platform, hanging latrines and bucket latrines
Limited	Sanitation facilities of an otherwise acceptable type shared between two or more households
Basic unsafe	A basic sanitation facility which is not shared with other households, but excreta are not disposed safely, such as flushed but not disposed to piped sewer system, septic tank or pit latrine
Improved	Sanitation facility which is not shared with other households and where excreta are safely disposed in situ or treated off-site and includes flush/pour flush to piped sewer system, septic tank or pit latrine, ventilated improved pit latrine, composting toilet or pit latrine with a slab

One target of the United Nations' sixth Sustainable Development Goal (whose aim is to "ensure availability and sustainable management of water and sanitation for all") is: "by 2030, [to] achieve access to adequate and equitable sanitation and hygiene for all and end open defecation." To achieve the target, the Joint Monitoring Programme (JMP) of the World Health Organisation and the UNICEF proposes using "a *service ladder approach* to benchmark and track progress across countries at different stages of development", building on the existing datasets.[10] We pursue this service ladder approach and apply our ordinal poverty measures to study the improvement in sanitation deprivation in Bangladesh. We

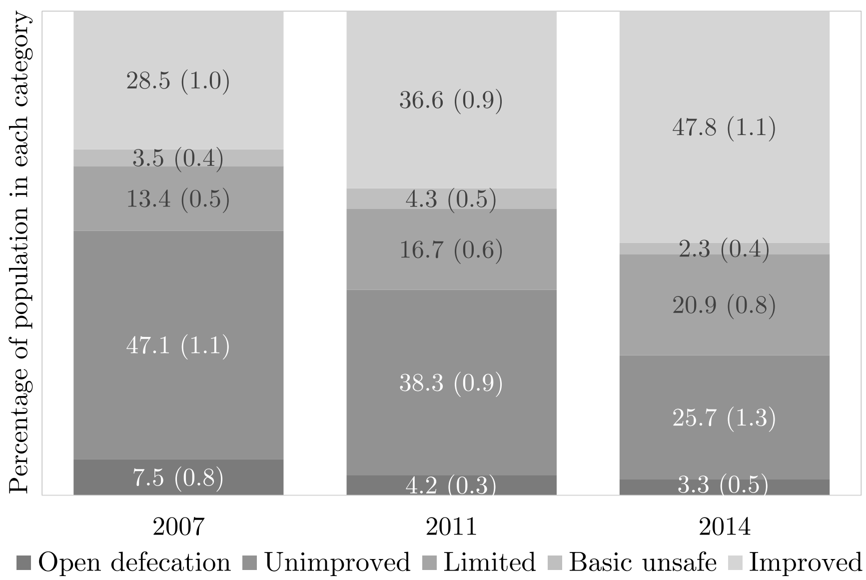
[8] Available at https://sustainabledevelopment.un.org/sdgs.

[9] See NIPORT et al. (2009, 2013, 2016) for details about the survey design.

[10] The JMP document titled *WASH Post-2015: Proposed indicators for drinking water, sanitation and hygiene* was accessed in April 2017 at https://www.wssinfo.org.

classify households' *access to sanitation* in five ordered categories presented in Table 1. The five categories are ordered as: 'open defecation' $\succ_D$ 'unimproved' $\succ_D$ 'limited' $\succ_D$ 'basic unsafe' $\succ_D$ 'improved'. We consider all persons living in a household deprived in access to sanitation if the household experiences any category other than the 'improved' category.

Figure 1: Change in population distribution across sanitation categories in Bangladesh



Source: Authors' own computations. Standard errors are reported in parentheses.

Figure 1 shows how the estimated population shares in different deprivation categories have evolved over time in Bangladesh. Clearly, the estimated percentage in the 'improved' category has gradually increased (statistically significantly) from 28.5% in 2007 to 36.6% in 2011 to 47.8% in 2014. Thus, the proportion of population in deprived categories has gone down over the same period. Changes within the deprived categories are however mixed. Although the estimated population shares in the two most deprived categories ('open defecation' and 'unimproved') have decreased (statistically significantly) systematically between 2007 and 2014, the population shares in the other two deprivation categories have not.

Has this estimated reduction pattern been replicated within all divisions? Table 2 presents the changes in the discrete probability distributions in three divisions: Dhaka, Rajshahi and Sylhet.[11] The estimated population shares in the 'improved' category have increased (statistically significantly) gradually in all three regions (Table 2), and so the shares of deprived population have gone down. We however point out two crucial aspects.

First, let us compare the reduction patterns in Dhaka and Rajshahi. The population share in the 'improved' category is higher in Rajshahi in 2011 and 2014 and statistically indistinguishable in 2007 implying that sanitation deprivation is never lower in Dhaka. However, the estimated population shares in the two most deprived categories ('open defecation' and

[11] A new division called Rangpur was formed in 2010, which was a part of the Rajshahi Division. The Rangpur division did not exist during the 2007 DHS survey and so we have combined this new division with the Rajshahi division in 2011 and 2014 DHS surveys to preserve comparability over time.

Table 2: Change in the population shares across sanitation categories in Dhaka, Rajshahi and Sylhet

	Dhaka			Rajshahi			Sylhet		
Category	2007	2011	2014	2007	2011	2014	2007	2011	2014
Open defecation	7.5	4.0	2.2	13.8	3.9	3.2	2.1	12.5	9.4
	(1.4)	(0.7)	(0.8)	(2.3)	(0.8)	(0.9)	(0.4)	(1.5)	(1.3)
Unimproved	44.3	35.9	22.6	45.3	36.4	28.3	57.2	34.3	23.0
	(1.9)	(1.7)	(2.8)	(2.6)	(3.1)	(2.7)	(3.3)	(2.0)	(2.6)
Limited	14.4	18.0	26.1	14.7	20.7	20.2	10.1	17.7	22.3
	(1.1)	(1.5)	(2.0)	(1.2)	(1.4)	(1.3)	(1.9)	(0.9)	(1.8)
Basic unsafe	8.6	10.5	5.4	0.2	0.2	0.3	0.6	0.1	0.3
	(1.0)	(1.6)	(1.0)	(0.1)	(0.1)	(0.1)	(0.2)	(0.1)	(0.2)
Improved	25.2	31.6	43.7	26.0	38.8	48.0	30.1	35.4	45.0
	(1.9)	(1.9)	(2.5)	(1.9)	(2.3)	(2.3)	(2.3)	(1.9)	(1.7)

Source: Authors' own computations. Standard errors are reported in parentheses.

'unimproved') are higher in Rajshahi than in Dhaka in 2007 and 2014 and statistically indistinguishable in 2011. Second, like Dhaka and Rajshahi in Table 2, sanitation deprivation in Sylhet has also improved gradually. However, the estimated population share in the poorest category ('open defecation') is significantly higher in 2011 and in 2014 than in 2007. A simple headcount measure, which only captures the proportion of the overall deprived population, would always overlook these substantial differences.

Table 3: Change in sanitation deprivation by ordinal poverty measures in Bangladesh and its divisions

	H			P_1			P_2			P_3		
Region	2007	2011	2014	2007	2011	2014	2007	2011	2014	2007	2011	2014
Barisal	66.1	60.5	46.8	47.7	44.0	32.6	35.0	32.6	23.4	25.2	23.5	16.7
	(2.7)	(2.1)	(3.3)	(1.9)	(1.7)	(2.9)	(1.4)	(1.4)	(2.4)	(1.1)	(1.1)	(1.8)
Chittagong	67.1	59.2	44.9	47.0	38.9	29.2	34.8	27.3	20.2	26.2	19.6	14.9
	(2.9)	(2.1)	(3.1)	(2.8)	(1.7)	(2.9)	(2.7)	(1.5)	(2.7)	(2.6)	(1.2)	(2.5)
Dhaka	74.8	68.4	56.3	50.1	42.6	33.5	36.6	29.4	21.8	27.8	21.6	15.4
	(1.9)	(1.9)	(2.5)	(1.7)	(1.3)	(1.8)	(1.6)	(1.1)	(1.6)	(1.4)	(1.0)	(1.3)
Khulna	69.0	61.4	50.3	48.9	41.8	33.0	35.7	29.4	22.7	25.9	20.9	16.1
	(1.8)	(1.6)	(2.3)	(1.4)	(1.2)	(1.7)	(1.1)	(1.0)	(1.4)	(1.0)	(0.7)	(1.0)
Rajshahi	74.0	61.2	52.0	55.2	41.6	34.7	43.0	29.6	24.2	34.1	21.6	17.6
	(1.9)	(2.3)	(2.3)	(1.8)	(1.9)	(1.8)	(1.9)	(1.6)	(1.5)	(1.9)	(1.3)	(1.3)
Sylhet	69.9	64.6	55.0	50.1	47.1	37.9	36.8	36.2	27.9	26.5	28.9	21.9
	(2.3)	(1.9)	(1.7)	(1.9)	(1.6)	(1.7)	(1.5)	(1.4)	(1.6)	(1.1)	(1.4)	(1.4)
Bangladesh	71.5	63.4	52.2	50.4	42.3	33.5	37.5	30.1	23.1	28.5	22.2	16.8
	(1.0)	(0.9)	(1.1)	(0.9)	(0.7)	(0.9)	(0.9)	(0.6)	(0.8)	(0.8)	(0.5)	(0.7)

Source: Authors' own computations. Standard errors are reported in parentheses.

Table 3 presents four different poverty measures for Bangladesh and for its six divisions. We assume the poverty threshold category to be 'basic unsafe'. The first poverty measure is the headcount ratio (H), which, in this context, is the population share experiencing any one of the four deprivation categories. The second measure is P_1, such that $P_1 \in \mathcal{P} \setminus \{\mathcal{P}_\alpha\}$ for $\alpha \geq 1$, and is defined by the ordering weights $\boldsymbol{\omega}^1 = (1, 0.75, 0.5, 0.25, 0)$. The third measure is $P_2 \in \mathcal{P}_1 \setminus \{\mathcal{P}_\alpha\}$ for $\alpha \geq 2$ with ordering weights $\boldsymbol{\omega}^2 = (1, 0.75^2, 0.5^2, 0.25^2, 0)$, i.e. respecting the restrictions in Corollary 4.1, but not respecting, for instance, the restrictions in Theorem 4.2 or the restrictions in Theorem 4.1 for $\alpha \geq 2$; whereas, the fourth measure is $P_3 \in \mathcal{P}_S$ with ordering weights $\boldsymbol{\omega}^3 = (1, 0.4, 0.15, 0.05, 0)$, i.e. respecting the restrictions in Theorem 4.2. Note that measures P_2 and P_3 provide precedences to those that are in the poorer categories. All four measures lie between 0 and 1, but we have multiplied them by hundred so that they lie between 0 (lowest deprivation) and 100 (highest deprivation).

Comparison of these measures provide useful insights; especially into the two crucial aspects that we have presented in Table 2. The headcount ratio estimate in Dhaka is statistically indistinguishable from the headcount ratio estimate in Rajshahi for 2007, despite deprivation in the two poorest categories being higher in Rajshahi. However, this crucial aspect is captured by the latter three measures, which show statistically significantly higher poverty estimates in Rajshahi than in Dhaka. Similarly, the headcount ratio estimate is higher in Dhaka than in Rajshahi for 2011, but the difference vanishes when poverty is assessed by the other three ordinal measures.

7. Application to multidimensional poverty measurement

So far we have focused on a single dimension. The literature on multidimensional poverty, however, has grown significantly over the last two decades and so has the surrounding debate. Several multidimensional poverty measures have been proposed in the literature since the seminal work of Chakravarty et al. (1998) and Bourguignon and Chakravarty (2003) under the assumption that the underlying dimensions are cardinal. Yet one important concern has been how to conduct meaningful poverty assessment when the underlying dimensions are ordinal. This challenge has not been overlooked and various multidimensional poverty measures motivated by the *counting approach* (Atkinson, 2003) have been proposed (see, for instance, Chakravarty and D'Ambrosio, 2006; Alkire and Foster, 2011; Aaberge and Peluso, 2012; Bossert et al., 2013; Alkire and Foster, 2016; Dhongde et al., 2016).

Multidimensional counting measures are based on simultaneous deprivations across different dimensions. Additively decomposable counting poverty measures are constructed in the following steps for a hypothetical society with N individuals and $\mathcal{D} \geq 2$ dimensions. First, if an individual n is deprived in dimension d, then the person is assigned a *deprivation status score* of $g_{nd} = 1$; whereas, the person is assigned a score of $g_{nd} = 0$, otherwise. The same goes for all $n = 1, \ldots, N$ and for all $d = 1, \ldots, \mathcal{D}$.

Second, a relative weight w_d is assigned to the d^{th} deprivation, such that $w_d > 0$ and $\sum_{d=1}^{\mathcal{D}} w_d = 1$, and an *attainment score* $\sigma_n = \sum_{d=1}^{\mathcal{D}} w_d(1 - g_{nd})$ is obtained for all $n = 1, \ldots, N$.[12] By construction, $0 \leq \sigma_n \leq 1$ for all n and a larger attainment score reflects

[12]Note that the relative weights w_d's assigned to dimensions are different from the ordering weights ω_s's.

lower level of deprivation. Let us denote the n attainment scores by $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$. Given that $\mathcal{D}$ is finite, each weighting choice generates a finite number of S attainment scores, in turn creating S categories $c_1, \ldots, c_S$, such that category c_S reflects least multiple deprivation and category c_1 reflects largest multiple deprivation. We denote the attainment score corresponding to category c_s by $\mathcal{C}_s$ so that whenever individual n experiences category c_s, $\sigma_n = \mathcal{C}_s$. We denote the proportion of population experiencing score $\mathcal{C}_s$ by $p_s(\boldsymbol{\sigma})$.

In the third step, a category c_k for any $k < S$ is selected as a poverty threshold category to identify the poor, such that all people experiencing category c_s for all $s \leq k$ are identified as poor. The additively decomposable counting measures (P^C) are expressed as:

$$P^C(\boldsymbol{\sigma}; c_k) = \sum_{s=1}^{S} f(\mathcal{C}_s) p_s(\boldsymbol{\sigma}); \tag{3}$$

where $f(\mathcal{C}_1) = 1$, $f(\mathcal{C}_s)$ is monotonically decreasing in its argument for all $s \leq k$, and $f(\mathcal{C}_s) = 0$ for all $s > k$.

Different measures use different functional forms of $f(\mathcal{C}_s)$. For example, Alkire and Foster (2011) use $f(\mathcal{C}_s) = (1 - \mathcal{C}_s)$ for all $s \leq k$; Chakravarty and D'Ambrosio (2006) use $f(\mathcal{C}_s) = (1 - \mathcal{C}_s)^\beta$ for $\beta \geq 1$ and for all $s \leq k = S - 1$; whereas Alkire and Foster (2016) use $f(\mathcal{C}_s) = (1 - \mathcal{C}_s)^\alpha$ for $\alpha \geq 1$ and for all $s \leq k$.[13] Note that Equations (2) and (3) are identical to each other, as the restrictions on $f(\mathcal{C}_s)$ are the same as the restrictions on ω_s for all s. Thus, the additively decomposable multidimensional counting measures can be expressed as the class of ordinal poverty measures in Theorem 3.1.

One controversial aspect surrounding the counting measures presented in Equation (3) is that they require assigning precise weight (i.e., w_d) to each dimensional deprivation (Ravallion, 2011; Ferreira and Lugo, 2013). If there is agreement about a set of precise weights, then indeed counting approaches are highly amicable to policy applications.[14] However, if there is a unanimous agreement only over the ordinal ranking of different combinations of deprivations, but dissent on the precise weights, then is a non-counting multidimensional approach feasible using the ordinal measurement method that we have developed?

A second source of controversy is that counting approaches for ordinal variables are not developed for capturing depth of deprivations within dimensions because each dimension is first dichotomised into sets of deprived and non-deprived people. Yet capturing depth of deprivations within dimensions may be of great policy interest.[15] Can a non-counting multidimensional approach be applied using our ordinal measurement method?

Our answer to both questions is *yes*, which can be demonstrated with an example. Suppose, poverty is assessed by using two dimensions: E and H, where there are two categories E_1, E_2 in dimension E and three categories H_1, H_2, H_3 in dimension H. Category E_1 in dimension E

[13] Alkire and Foster (2016) use the parameter values of $\alpha \geq 0$, but we ignore the value of $\alpha = 0$ as the measure becomes the multidimensional headcount ratio.

[14] For discussions on weights in the counting approach framework, see Alkire et al. (2015, Chapter 6).

[15] For an approach to identification (but not aggregation) using a depth approach versus using an intensity approach in counting framework, see Alkire and Seth (2016).

and categories H_1 and H_2 in dimension H reflect deprivations. Suppose we have the following information about the ordering of deprivations: (i) $E_1 \succ_D E_2$, (ii) $H_1 \succ_D H_2 \succ_D H_3$, and (iii) $H_2 \succ_D E_1$. The first two conditions present the ordering within each dimension; whereas the third conveys that a single deprivation in any category of dimension H is worse than a single deprivation in dimension E. Here, we are only aware of the ordinal ranking of deprivation categories within as well as across two dimensions.

In the multidimensional context, poverty is a reflection of different combinations of deprivations in different dimensions obtained through an identification function. The three aforementioned restrictions, along with the additional restriction that multiple deprivations are worse than a single deprivation, leads to the following raking among the poor:

$$(E_1, H_1) \succ_D (E_1, H_2) \succ_D (E_2, H_1) \succ_D (E_2, H_2) \succ_D (E_1, H_3) \succ_D (E_2, H_3).$$

Thus, there are now six ordered categories (i.e., $S = 6$). Under the assumption that the poverty threshold category is $c_k = (E_2, H_2)$, a person must be deprived in dimension H in order to be identified as poor, leaving four ordered poverty categories ($k = 4$). Clearly, Theorem 3.1 as well as the concept of *precedence to poorer people* are applicable to this case. This example may be easily extended to cases involving more than two dimensions.

Notice that neither did we consider any precise set of weights, nor dichotomise all dimensions. Dimension H rather had more than one deprivation category. Hence, not only we argue that the multidimensional counting approaches can be expressed as the ordinal poverty measures in Theorem 3.1, but we also argue that our ordinal poverty measures may be used for a broader and holistic multidimensional framework.

8. Concluding remarks

There is little doubt that poverty is a multidimensional concept and the current global development agenda correctly seeks to "reduce poverty in all its dimension". To meet this target, it is indeed important to assess poverty from a multidimensional perspective. However, one should not discredit the potential interest for evaluating the impact of a targeted program in reducing deprivation in a single dimension such as educational or health outcomes and access to public services, which may often be assessed by an ordinal variable with multiple ordered deprivation categories. The frequently used headcount ratio, in this case, is ineffective as it overlooks the depth of deprivations, i.e., any changes within the ordered deprivation categories.

Our paper has thus posed the question: "how should we assess poverty when variables are ordinal?" Implicitly, the companion question is: "Can we meaningfully assess poverty going beyond the headcount ratio when we have an ordinal variable?" Drawing on eight reasonable axiomatic properties, our answer is: "Poverty can be measured with ordinal variables through weighted averages of the discrete probabilities corresponding to the ordered categories." We refer to these weights as ordering weights, which need to satisfy a specific set of restrictions in order to render the social poverty indices in fulfillment of the key properties. Our axiomatically characterised class of social poverty indices has certain desirable features, such as additive decomposability and being bounded between zero (when none experiences any deprivation) and one (when everyone experiences the most deprived category). Remarkably, these desirable features are not mere assumptions, but are logical consequences stemming from our eight axioms.

By contrast to previous attempts in the literature on poverty measurement with ordinal variables, we have gone significantly further in the direction of operationalising different concepts of 'precedence to the poorer people among the poor', which ensures that the policymaker has incentive to assist the poorer over the less poor. We have shown that it is possible to devise reasonable poverty measures prioritising welfare improvements among the most deprived when variables are ordinal. We have axiomatically characterised a set of subclasses of ordinal poverty measures based on a continuum of different notions of precedence to the poorer among the poor. Each subclass is defined by an additional restriction on the admissible ordering weights. The precedence-sensitive measures have been proven useful in the illustration pertaining to sanitation deprivation in Bangladesh by highlighting those provinces where the overall headcount improvement did not come about through reductions in the proportion of population in the most deprived categories.

Since several poverty measures are admissible within each characterized class and subclasses, we have also developed stochastic dominance conditions for each subclass of poverty measures. Their fulfilment guarantees that all measures within a given class (or subclass) rank the same pair of distributions robustly. While some of these conditions represent the ordinal-variable analog of existing conditions for continuous variables in the poverty dominance literature (Foster and Shorrocks, 1988b), others are themselves a novel methodological contribution to the literature on stochastic dominance with ordinal variables, to the best of our knowledge.

Considering the recent surge in the literature on multidimensional poverty measurement, especially in the counting approach, we showed how our method is closely aligned with the aggregation procedure characteristic of the counting framework. It is still a usual practice to dichotomise deprivations within each dimension when using existing counting measures, ignoring the depth within each category. Future research could focus on how to further develop the counting measures in order to incorporate depth of deprivations in an ordinal framework.

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Appendices

Appendix A1. Proof of Theorem 3.1

It is straightforward to check that the poverty measures in Equation 2 satisfy properties OMN, ORC, CHR, and FOC. We now prove the necessary part that if a poverty measure in Lemma 3.1 satisfies properties OMN, ORC, CHR, and FOC, then it takes the functional form in Equation 2.

By Lemma 3.1, we already know that:

$$P(\mathbf{x}; c_k) = F\left[\frac{1}{N}\sum_{n=1}^{N}\phi(x_n)\right] \tag{A1}$$

where F is continuous and increasing and ϕ is continuous.

Now consider two singleton societies: $\mathbf{z} = (x_i)$ and $\mathbf{y} = (x_j)$, such that $i \in \Omega_s(\mathbf{z})$ and $j \in \Omega_s(\mathbf{y})$ for some $s \in \{1, \dots, S\}$. By property ORC, we already know that $P(\mathbf{z}; c_k) = P(\mathbf{y}; c_k)$. Therefore, from Equation A1, we obtain:

$$F(\phi(x_i)) = F(\phi(x_j)).$$

Given that F is increasing and continuous, we must have $\phi(x_i) = \phi(x_j)$. So, if $\mathbf{x} = (x_i, x_j)$ such that $i, j \in \Omega_s(\mathbf{x})$, it then follows that $\phi(x_i) = \phi(x_j)$. Clearly, thus, for any $\mathbf{x} \in \mathbf{X}$ and for any $n, n' \in \mathbf{N}(\mathbf{x})$, if $n, n' \in \Omega_s(\mathbf{x})$ for some $s \in \{1, \dots, S\}$, then $\phi(x_n) = \phi(x_{n'})$. Let us denote $\phi(x_n) = u_s$ whenever $n \in \Omega_s$ for any $s \in \{1, \dots, S\}$ and for any $n \in \mathbf{N}(\mathbf{x})$. We can now rewrite Equation A1 as:

$$P(\mathbf{x}; c_k) = F\left[\sum_{s=1}^{S} u_s p_s(\mathbf{x})\right] \tag{A2}$$

Now, consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$, such that $n \in \Omega_s(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ for all $n \in \mathbf{N}(\mathbf{x})$, but $n \in \Omega_{s'}(\mathbf{y})$ for all $n \in \mathbf{N}(\mathbf{x})$ for some $s' > s$. Note in this case that $p_s(\mathbf{x}) = p_{s'}(\mathbf{y}) = 1$. Then, from property OMN, we know that $P(\mathbf{y}; c_k) < P(\mathbf{x}; c_k)$. Therefore, using Equation A2, we obtain:

$$F[u_{s'}] < F[u_s].$$

Given that F is increasing, we get:

$$u_{s'} < u_s. \tag{A3}$$

The relationship in Equation A3 holds for any s, such that $s \leq k$ but $s < s'$. In other words, $u_{s-1} > u_s > u_{s'}$ for all $s = 2, ..., k$ and for any $s' > k$, whenever $k \geq 2$. When $S = 2$, then $k = 1$ and so $u_1 > u_2$.

We next use property CHR. Suppose, $k = 1$. Then, by property CHR, Equation A2 yields:

$$F[p_1(\mathbf{x})u_1] = p_1(\mathbf{x}). \tag{A4}$$

Note, by definition, that $0 \leq p_1(\mathbf{x}) \leq 1$. Hence, $F(0) = 0$ and $F(u_1) = 1$. Assume, $q = p_1(\mathbf{x})u_1$. Expressing Equation A4 in terms of q yields:

$$F(q) = \frac{q}{u_1}. \tag{A5}$$

Combining Equations A2 and A5, we obtain:

$$P(\mathbf{x}; c_k) = \sum_{s=1}^{S} p_s(\mathbf{x}) \frac{u_s}{u_1}. \tag{A6}$$

Substituting $\omega_s = u_s/u_1$ for all $s = 1, ..., k$ in Equation A6, we arrive at:

$$P(\mathbf{x}; c_k) = \sum_{s=1}^{S} p_s(\mathbf{x})\omega_s. \tag{A7}$$

Clearly, $\omega_1 = u_1/u_1 = 1$. Moreover, from Equation A3, it follows that $\omega_{s-1} > \omega_s > \omega_{s'}$ for all $s = 2, ..., k$ and for any $s' > k$ whenever $k \geq 2$.

In order to complete the proof, we need to show that $\omega_s = 0$ for all $s > k$. For this purpose, consider any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_N$, such that $n \in \Omega_{s'}(\mathbf{x}) \subseteq \mathbf{Z}^{NP}(\mathbf{x}; c_k)$ for all $n \in \mathbf{N}(\mathbf{x})$ and $n \in \Omega_{s''}(\mathbf{y}) \in \mathbf{Z}^{NP}(\mathbf{y}; c_k)$ for all $n \in \mathbf{N}(\mathbf{y})$ for any $s'' > s' > k$. Note that $p_{s'}(\mathbf{x}) = p_{s''}(\mathbf{y}) = 1$ and indeed $p_s(\mathbf{x}) = p_s(\mathbf{y}) = 0 \; \forall s \leq k$. By property FOC, we then require $P(\mathbf{y}; c_k) = P(\mathbf{x}; c_k)$. Thus, from Equation A7, we obtain $P(\mathbf{x}; c_k) = \omega_s = \omega_{s'} = P(\mathbf{y}; c_k)$ for any $s' > s > k$. Since, $p_s(\mathbf{x}) = p_s(\mathbf{y}) = 0 \; \forall s \leq k$, it follows that $p_1(\mathbf{x}) = p_1(\mathbf{y}) = 0$. Consider $k = 1$. Then, by property CHR, we must have $P(\mathbf{x}; c_k) = P(\mathbf{y}; c_k) = 0$. Hence, it must be the case that: $\omega_s = 0$ for all $s > k$, which completes our proof. ■

Appendix A2. Proof of Theorem 4.1

The sufficiency part is straightforward. We prove the necessity part as follows.

Suppose $k \geq 2$ and $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq k - 1$. Now, suppose, $\mathbf{y}$ and $\mathbf{z}$ are obtained from $\mathbf{x} \in \mathbf{X}_N$ as follows: For some $n'' \neq n'$ and some $t > s'$, $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_{s'}(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{s'+1}(\mathbf{y})$, while $x_n = y_n$ for all $n \neq n'$; whereas, $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n'' \in \Omega_{t'}(\mathbf{z})$ for some $t > s'$ and $t' = \min\{t + \alpha, S\}$, while $x_n = z_n$ for all $n \neq n''$. It follows that $\Omega_s(\mathbf{y}) = \Omega_s(\mathbf{x})$ for all $s \neq s', s' + 1$ and $\Omega_s(\mathbf{z}) = \Omega_s(\mathbf{x})$ for all $s \neq t, t'$; whereas $\Omega_{s'}(\mathbf{y}) = \Omega_{s'}(\mathbf{x}) - 1$, $\Omega_{s'+1}(\mathbf{y}) = \Omega_{s'+1}(\mathbf{x}) + 1$, $\Omega_t(\mathbf{z}) = \Omega_t(\mathbf{x}) - 1$, and $\Omega_{t'}(\mathbf{z}) = \Omega_{t'}(\mathbf{x}) + 1$. Note that by construction $t \leq k$.

By property PRE-α, we know that:

$$P(\mathbf{y}; c_k) - P(\mathbf{z}; c_k) < 0. \tag{A8}$$

Combining Equations 2 and A8, we get:

$$\omega_{s'+1} - \omega_{s'} - \omega_{t'} + \omega_t < 0.$$

Substituting $t = s' + 1 = s$ for any $s = 2, \ldots, k$, we obtain:

$$\omega_{s-1} - \omega_s > \omega_s - \omega_{t'}. \tag{A9}$$

First, suppose $t' = s + \alpha \leq k < S$ or $s \leq k - \alpha$. Then $\omega_{t'} = \omega_{s+\alpha} > 0$ by Theorem 3.1 and Equation A9 can be expressed as $\omega_{s-1} - \omega_s > \omega_s - \omega_{s+\alpha}$ for all $s = 2, \ldots, k - \alpha$. Second, suppose $t' = \min\{s + \alpha, S\} > k$ or $s > k - \alpha$. We know that $\omega_s = 0$ for all $s > k$ by Theorem 3.1 and so Equation A9 can be expressed as $\omega_{s-1} - \omega_s > \omega_s$ or $\omega_{s-1} > 2\omega_s$ for all $s = k - \alpha + 1, \ldots, k$. This completes the proof.

■

Appendix A3. Proof of Theorem 4.2

Let us first prove the sufficiency part. Suppose $k \geq 2$. We already know from Theorem 3.1 that $\omega_{s-1} > \omega_s > 0$ for all $s = 2, \ldots, k$ and $\omega_s = 0$ for all $s > k$. Suppose additionally that $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$. Alternatively, $\omega_{s-1} - \omega_s > \omega_s$ for all $s = 2, \ldots, k$.

For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}_N$, for any $c_k \in \mathbf{C}_{-S}$, and for some $n'' \neq n'$, suppose $\mathbf{y}$ is obtained from $\mathbf{x}$, such that $n' \in \Omega_v(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ but $n' \in \Omega_{v+\alpha}(\mathbf{y})$ for some $\alpha \in \mathbb{N}$, while $x_n = y_n$ for all $n \neq n'$, and $\mathbf{z}$ is obtained from $\mathbf{x}$, such that $n'' \in \Omega_t(\mathbf{x}) \subseteq \mathbf{Z}^P(\mathbf{x}; c_k)$ for some $t \geq v + \alpha$ but $n'' \in \Omega_{t+\beta}(\mathbf{z})$ for some $\beta \in \mathbb{N}$, while $x_n = z_n$ for all $n \neq n''$. By definition, $t \leq k$.

It follows that $\Omega_s(\mathbf{y}) = \Omega_s(\mathbf{x})$ for all $s \neq v, v{+}\alpha$ and $\Omega_s(\mathbf{z}) = \Omega_s(\mathbf{x})$ for all $s \neq t, t{+}\beta$; whereas $\Omega_v(\mathbf{y}) = \Omega_v(\mathbf{x}) - 1$, $\Omega_{v+\alpha}(\mathbf{y}) = \Omega_{v+\alpha}(\mathbf{x}) + 1$, $\Omega_t(\mathbf{z}) = \Omega_t(\mathbf{x}) - 1$, and $\Omega_{t+\beta}(\mathbf{z}) = \Omega_{t+\beta}(\mathbf{x}) + 1$. With the help of Equation 2, we get:

$$P(\mathbf{y}; c_k) - P(\mathbf{z}; c_k) = \omega_{v+\alpha} - \omega_v - \omega_{t+\beta} + \omega_t = (\omega_t - \omega_{t+\beta}) - (\omega_v - \omega_{v+\alpha}). \tag{A10}$$

By assumption of the sufficiency part: $\omega_{s-1} - \omega_s > \omega_s$ for all $s = 2, \ldots, k$. Combining this assumption with the weight restrictions from Theorem 3.1 we can easily deduce that: $(\omega_v - \omega_{v+\alpha}) > (\omega_v - \omega_{v+1}) > \omega_{v+1} > \omega_{v+\alpha}$. Hence: $(\omega_v - \omega_{v+\alpha}) > \omega_{v+\alpha}$. Since $v + \alpha \leq t \leq k$ and $\omega_{s-1} > \omega_s > 0$ for all $s = 2, \ldots, k$, it also follows that $\omega_{v+\alpha} \geq (\omega_t - \omega_{t+\beta})$. Hence, $(\omega_v - \omega_{v+\alpha}) > (\omega_t - \omega_{t+\beta})$ and $P(\mathbf{y}; c_k) < P(\mathbf{z}; c_k)$.

We next prove the necessity part starting with Equation A10. By property PRE-S, we know that $P(\mathbf{y}; c_k) < P(\mathbf{z}; c_k)$. Thus:

$$\omega_v - \omega_{v+\alpha} > \omega_t - \omega_{t+\beta}. \tag{A11}$$

Now the inequality in Equation A11 must hold for any situation in which $t \geq v+\alpha$, including the comparison of the minimum possible improvement for the poorer person, given by $\omega_v - \omega_{v+1}$ (i.e. with $\alpha = 1$), against the maximum possible improvement for the less poor person, given by $\omega_t - \omega_{t+\beta}$ with $t = v + 1$ and $t + \beta > k$. Inserting these values into Equation A11, bearing in mind that $\omega_{t+\beta} = 0$ when $t + \beta > k$, yields:

$$\omega_v - \omega_{v+1} > \omega_{v+1}.$$

Substituting $v = s - 1$ for any $s = 2, \ldots, k$ yields $\omega_{s-1} - \omega_s > \omega_s$. Hence, $\omega_{s-1} > 2\omega_s$ for all $s = 2, \ldots, k$. ■

Appendix A4. Proof of Theorem 5.1

We first prove the sufficiency part. From Theorem 3.1, we know that $\omega_s = 0$ for all $s > k$. Thus, Equation 2 may be presented using the difference operator as $\Delta P_k = \sum_{s=1}^{k} \omega_s \Delta p_s$. Using summation by parts, also known as Abel's lemma (Guenther and Lee, 1988), it follows that:

$$\Delta P_k = \sum_{s=1}^{k-1} [\omega_s - \omega_{s+1}] \Delta F_s + \Delta F_k \omega_k. \tag{A12}$$

We already know from Theorem 3.1 that $\omega_k > 0$ and $\omega_s - \omega_{s+1} > 0$ $\forall s = 1, \ldots, k-1$. Therefore, clearly from equation A12, the condition that $\Delta F_s \leq 0$ for all $s \leq k$ and $\Delta F_s < 0$ for at least one $s \leq k$ is sufficient to ensure that $\Delta P_k < 0$ for all $P \in \mathcal{P}$ and for a given $c_k \in \mathbf{C_{-S}}$.

We next prove the necessity part by contradiction. Either, consider the situation, where $\Delta F_t > 0$ for some $t \leq k$, $\Delta F_s \leq 0$ for all $s \leq k$ but $s \neq t$, and $\Delta F_s < 0$ for some $s \leq k$ but $s \neq t$. For a sufficiently large value of $\omega_t - \omega_{t+1}$ in Equation A12, it may always be possible that $\Delta P_k > 0$. Or, consider the situation $\Delta F_s = 0$ for all $s \leq k$. In this case, $\Delta P_k = 0$. Hence, the necessary condition requires both $\Delta F_s \leq 0$ for all $s \leq k$ and $\Delta F_s < 0$ for some $s \leq k$. This completes the proof. ■

Appendix A5. Proof of Theorem 5.2

Summing by parts the first component on the right-hand side of Equation A12 yields:

$$\Delta P_k = \sum_{s=1}^{k-2} \left(\{[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}]\} \sum_{\ell=1}^{s} \Delta F_\ell \right) + [\omega_{k-1} - \omega_k] \sum_{s=1}^{k-1} \Delta F_s + \Delta F_k \omega_k. \tag{A13}$$

a. *Sufficiency*: Define $\lambda_s(\alpha) = (\omega_{s-1} - 2\omega_s + \omega_{s+\alpha}) + (\omega_{s+1} - \omega_{s+\alpha})$ for all $s = 2, \ldots, k-\alpha$ and $\eta_s(\alpha) = (\omega_{s-1} - 2\omega_s) + \omega_{s+1}$ for all $s = k - \alpha + 1, \ldots, k - 1$. Then the first component in Equation A13 can be decomposed into two components and the last two components may be rearranged to rewrite the equation as:

$$\begin{aligned}\Delta P_k = &\sum_{s=2}^{k-\alpha}\left(\lambda_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k-1}\left(\eta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) \\ &+ (\omega_{k-1} - 2\omega_k)\sum_{s=1}^{k-1}\Delta F_s + \omega_k\sum_{s=1}^{k}\Delta F_s.\end{aligned} \tag{A14}$$

Given that $\omega_{s-1} - 2\omega_s + \omega_{s+\alpha} > 0$ for all $s = 2, \ldots, k-\alpha$ by Theorem 4.1 and $\omega_{s+1} - \omega_{s+\alpha} \geq 0$ for any $1 \leq \alpha \leq k-1$ by Theorem 3.1, we know that $\lambda_s(\alpha) > 0$ for all $s = 2, \ldots, k-\alpha$. Similarly, given that $\omega_{s-1} - 2\omega_s > 0$ for all $s = k-\alpha+1, \ldots, k-1$ by Theorem 4.1 and $\omega_{s+1} \geq 0$ by Theorem 1, we know that $\eta_s(\alpha) > 0$ for all $s = k-\alpha+1, \ldots, k-1$. We further know that $\omega_{k-1} - 2\omega_k > 0$ by Theorem 4.1 and that $\omega_k > 0$ by Theorem 3.1. It is now straightforward to check from Equation A14 that $\sum_{\ell=1}^{s}\Delta F_\ell \leq 0\ \forall s \leq k$ and $\sum_{\ell=1}^{s}\Delta F_\ell < 0$ for at least one $s \leq k$, suffice for $\Delta P_k < 0$ for a given $c_k \in \mathbf{C_{-S}}$.

b.i. *Necessity when* $\alpha = 1$: We can rewrite equation A13 the following way:

$$\Delta P_k = \sum_{s=1}^{k-1}\left(\{[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}]\}\sum_{\ell=1}^{s}\Delta F_\ell\right) + \omega_k\sum_{s=1}^{k}\Delta F_s. \tag{A15}$$

We know from Theorem 4.1 that $[\omega_s - \omega_{s+1}] - [\omega_{s+1} - \omega_{s+2}] > 0\ \forall s = 1, ..., k-1$. Likewise $\omega_k > 0$. Yet we do not have any further restriction stating whether any of the weight functions in equation A15 is strictly greater than the others. Therefore every sum of cumulatives $(\sum_{\ell=1}^{s}\Delta F_\ell,\ s = 1, ..., k)$ must be non-positive and at least one strictly negative in order to ensure $\Delta P_k < 0$.

b.ii. *Necessity when* $2 \leq \alpha \leq k-1$: Define additionally $\delta_s(\alpha) = (\omega_{s-1} - 2\omega_s)$, such that $\delta_s(\alpha) = \eta_s(\alpha) - \omega_{s+1}$, for all $s = k-\alpha+1, \ldots, k-1$. Then, the middle two components of Equation A14 may be rearranged to rewrite the equation as:

$$\begin{aligned}\Delta P_k = &\sum_{s=2}^{k-\alpha}\left(\lambda_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k-\alpha+2}\left(\delta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) \\ &+ \sum_{s=k-\alpha+3}^{k}\left(\delta_s(\alpha)\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \sum_{s=k-\alpha+1}^{k}\left(\omega_{s+1}\sum_{\ell=1}^{s-1}\Delta F_\ell\right) + \omega_k\sum_{s=1}^{k}\Delta F_s.\end{aligned} \tag{A16}$$

We already know that $\lambda_s(\alpha) > 0$ for all $s = 2, \ldots, k-\alpha$ by Theorem 4.1 and Theorem 3.1 and it turns out that any of these weights can be larger than the other components' weights, it is thus necessary to have $\sum_{\ell=1}^{s}\Delta F_\ell \leq 0$ for all $s = 1, \ldots, k-\alpha-1$. Next, we also know that $\delta_s(\alpha) > 0$ for all $s = k-\alpha+1, \ldots, k-1$. However, not all $\delta_s(\alpha)$'s are necessarily larger than other component weights. Whenever $k-\alpha+3 \leq k$, it turns out that $\omega_{s-1} > \delta_s = \omega_{s-1} - 2\omega_s$ for all $s = k-\alpha+2, \ldots, k$. Therefore, even when the third component in Equation A16 is positive, it is possible to have $\Delta P_k < 0$ whenever the final two components are negative and carry sufficiently larger relative weight, given that the first two components are not positive. Hence, it is necessary that the final three components are jointly non-positive, provided largest possible weights are assigned to the final two components, which requires $\omega_s \to \omega_{s-1}/2$ for all $s = k - \alpha + 3, \ldots, k$. Consequently, $\delta_s \to 0$ for all $s = k - \alpha + 3, \ldots, k$. We use these conditions in the final three components of Equation A16 to obtain:

$$\begin{aligned}\Delta P_k = \sum_{s=2}^{k-\alpha} \left(\lambda_s \left(\alpha \right) \sum_{\ell=1}^{s-1} \Delta F_\ell \right) + \sum_{s=k-\alpha+1}^{k-\alpha+3} \left(\delta_s \left(\alpha \right) \sum_{\ell=1}^{s-1} \Delta F_\ell \right) \\ \omega_{k-\alpha+2} \left[\left(\sum_{\ell=1}^{k-\alpha} \Delta F_\ell \right) + \left(\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F_\ell \right) + 2^{1-\alpha} \Delta F_k \right].\end{aligned} \tag{A17}$$

Given that $\omega_{k-\alpha+2}$ may be higher than weights of the rest of the components, it is necessary that the third component be not positive. Hence, it is necessary that $\sum_{\ell=1}^{s} \Delta F_\ell \leq 0$ for all $s \leq k - \alpha + 1$ and $(\sum_{\ell=1}^{k-\alpha} \Delta F_\ell) + (\sum_{\ell=k-\alpha+1}^{k-1} 2^{k-\alpha-\ell} \Delta F_\ell) + 2^{1-\alpha} \Delta F_k \leq 0$ with at least one strict inequality for having $\Delta P_k < 0$. This completes the proof. ■