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### **Strategic fire-sales and price-mediated contagion in th banking system**

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# Strategic fire-sales and price-mediated contagion in the banking system

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## Abstract

We consider a price-mediated contagion framework in which each bank, after an exogenous shock, may have to sell assets in order to comply with regulatory constraints. Interaction between banks takes place only through price impact. We first characterize the equilibrium of the strategic fire sales problem and define measures of contagion. We then calibrate our model to publicly-available data – the US banks that were part of the 2015 regulatory stress-tests – and quantify contagion effects. We finally show how our framework may be used to draw regulatory measures such as the systemic risk capital surcharge for large banks.

**Keywords:** Fire sales, price-mediated contagion, Nash equilibrium with strategic complementarities, CCAR 2015, macro-prudential stress-tests

## 1 Introduction

Past financial crises have repeatedly shed light on the critical role played by financial institutions in propagating and amplifying an exogenous adverse shock [Brunnermeier, 2009, Krishnamurthy, 2010, Glasserman and Young, 2016]. This was recently illustrated during

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the 2007 subprime crisis when a shock in a relatively small asset class, the US subprime mortgages, resulted in magnified losses for numerous financial institutions due to contagion effects. A salient feature of the 2007 crisis is the role played by indirect, rather than direct, contagion effects [Clerc et al., 2016].

Direct contagion is the result of contractual links between financial institutions, typically debt or OTC derivatives: the failure of a given institution will trigger losses for its counterparties, potentially causing the defaults of other institutions, which will in turn trigger losses for their own counterparties and further failures etc... This direct contagion, generated by counterparty risk, has long been acknowledged as an important source of financial instability and has been studied by academics through network models ([Eisenberg and Noe, 2001, Elsinger et al., 2006, Fouque and Langsam, 2013, Upper and Worms, 2004]). Regulators have recently (partly) tackled counterparty risk by introducing collateral requirements and limitations of large exposures for OTC derivatives trades [Glasserman and Young, 2015].

Indirect (or price-mediated) contagion is in some sense a more subtle form of contagion as it occurs through price effects, even in the absence of direct contractual links between institutions: a given financial institution may be forced to sell some assets, pushing prices down and generating losses for all institutions holding the same assets. Such forced sales are generally referred to as *fire sales* and typically occur at a dislocated price when a distressed institution is willing to promptly liquidate part of its portfolio [Diamond and Rajan, 2011, Shleifer and Vishny, 2011]. Depending on the type of financial institution (i.e., depository institution, mutual fund, pension fund, insurance company, hedge fund ...), various reasons may be found to explain these fire sales, for instance collateralized short term financing [Shleifer and Vishny, 2011]. In the case of (insured) depository institutions, everything else equal, fire sales may indeed be triggered by regulatory capital requirements themselves.

One of the common features of the successive Basel regulations is that supervisors consider the following capital ratio to assess the solvency of banks:

$$\text{Risk based capital ratio} := \frac{\text{Total capital}}{\text{Risk Weighed Assets}} \quad (1)$$

Equation (1) is called a risk-based capital ratio (RBC) because its denominator is the total risk-weighted assets (RWA), defined as the sum of a few risk-related RWAs (typically credit, market and operational risks) rather than the total value of assets. In Basel III, the numerator of the RBC is defined as the sum of two types of capital, Tier 1 and Tier

2, where Tier 1 is designed to absorb losses without affecting the business as usual while Tier 2 is designed to absorb losses in case of liquidation of the bank<sup>1</sup>. Basel III regulation imposes that the RBC of a given bank must be *at least 8% at all times*. This means that when the assets of a bank are hit by an adverse shock and the bank's RBC drops below 8%, because it is generally too costly to issue new stocks in such a situation – typically due to the classical debt overhang problem [Hanson et al., 2011] and/or the adverse selection problem [Greenlaw et al., 2012] – the bank is likely to try to restore its capital ratio above 8% by selling assets, that is, by engaging fire sales. Such undesirable consequences of capital requirements that may induce banks to engage in fire sales have been recognized by academics and regulators. For example, [French et al., 2010, p46] note in their well-known collective report on the financial system that

“because of the mark to market accounting, fire sales by some firms may force others to liquidate positions to satisfy capital requirements. These successive sales can magnify the original temporary price drop and force more sales.”

while the Basel committee acknowledges (see [BCBS, 2014]) that during the subprime crisis, the banking sector was forced to

“reduce its leverage in a manner that amplified downward pressures on asset prices. This deleveraging process exacerbated the feedback loop between losses, falling bank capital and shrinking credit availability.”

While fire sales and price-mediated contagion appear to play a crucial role in spreading and amplifying market shocks, the academic literature on the topic, which is less abundant compared to that on direct contagion, either analyze past fire sales episodes instead of anticipating new fire sales ([Anton and Polk, 2014, Jotikasthira et al., 2012, Khandani and Lo, 2011, Cont and Wagalath, 2016]) or consider a simple (i.e., unweighted) capital ratio instead of a risk-based capital ratio for banks ([Caccioli et al., 2014, Greenwood et al., 2015]). We thus believe that there is still a need for a risk-based capital ratio model of indirect contagion that could be *easily calibrated* to public data (i.e., contained in the annual reports of banks) to anticipate fire sales in the banking system and its consequences after a common shock. Such a framework should be of interest for regulators as a possible toolkit to draw quantitative

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<sup>1</sup>See [BCBS, 2011], the official document on Basel III written by the Basel Committee on Banking Supervision (BCBS).

regulatory measures such as the systemic risk capital surcharge for large banks. The seminal paper by [Greenwood et al., 2015] started to bridge this gap by proposing such a framework that can be calibrated to public data<sup>2</sup>. However, they consider a simple capital ratio, a proxy for the leverage ratio and do not address the equilibrium (fire sales) problem.

In this paper, we explicitly consider an equilibrium model of strategic interaction through price-mediated contagion only in that each bank is assumed to hold the same risky marketable security such as an exchange-traded fund. The failure of one bank thus has no direct impact on the rest of the banking system as there is no contractual link (e.g., OTC derivatives, repo...) between banks. Contagion of failures may occur but only through a price effect caused by fire sales. In the literature on the subject, as already said, it is common to assume that banks are subject to simple capital ratio instead of a RBC but also that they implement simple rules of thumbs when liquidating their assets (e.g., [Caccioli et al., 2014], [Cont and Schaanning, 2016], [Greenwood et al., 2015]). In [Greenwood et al., 2015], it is assumed that each bank has a leverage target so that the bank trades assets when the leverage differs from the specified target. The authors note (in appendix B) that when the price impact is large enough, banks will be wiped out in a few periods but the equilibrium problem (i.e., the stationary state) is not considered. In the same vein, in [Capponi and Larsson, 2015], each bank tracks a fixed leverage target and buys or sells assets according to this objective. We depart from these papers in that we assume that banks are subject to a risk-based capital requirement, as implemented in practice by regulators and that they liquidate *optimally* their assets, as in [Cifuentes et al., 2005] or in [Braouezec and Wagalath, 2016]. More importantly, we recognize the *strategic aspect* of the liquidation problem. When some banks liquidate a non negligible portion of their assets, this generates a "negative externality" to the other market participants through the price impact and we consider, as usual in economics, the equilibrium situation in which no bank wants to unilaterally deviate from its equilibrium selling strategy, i.e., the Nash equilibrium. Our model of fire sales actually gives rise to a *game with strategic complementarities*, initiated in economic theory by [Milgrom and Roberts, 1990] and [Vives, 1990]). It turns out to be closely related to the one of [Cifuentes et al., 2005] in which they also consider an equilibrium situation but in a non-strategic framework. However, as they consider a network model à la [Eisenberg and Noe, 2001], its calibration remains dif-

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<sup>2</sup>Technically, they use public data from the European Banking Authority, which supplement the one contained in annual reports of banks.

difficult because information on bilateral exposures between banks are scarce [Upper, 2011], not to say unobservable. Within our framework, we make the simplifying assumption that there are no bilateral exposures between banks, that is, the unique source of systemic risk is price mediated contagion, but this allows us to easily calibrate our model to publicly-available data. Using the panel of banks considered by the American regulator to implement the 2015 regulatory stress tests (CCAR), once the parameters of each bank have been calibrated<sup>3</sup>, we compute the Nash equilibrium associated to the game of liquidation under various scenarios (shock/price impact) and quantify the effects of price-mediated contagion in terms of insolvency, i.e., the fraction of banks that are insolvent at equilibrium. In general, the relation between the price impact (or the shock) and the fraction of insolvent banks is *non-linear* and we quantify this non-linearity for the panel of banks under consideration. For instance, when the common shock is equal to 5%, at equilibrium, the fraction of insolvent banks is equal to 10% with a price impact of 3% while this fraction skyrockets to 30% with a price of 5%. We also make use of our model to evaluate the capital surcharge applicable to banks classified as Global Systemically Important Banks (GSIBs) phased in from 2016 and fully implemented in 2019. Quite interestingly, we show that when the common shock is equal to 6%, for a reasonable price impact, at equilibrium, seven banks are insolvent and six of them have been later on been classified as GSIBs. We then evaluate the efficiency of this higher loss absorbency (HLA) requirement by performing the (fictitious) exercise in which it is assumed that this additional capital surcharge is implemented as of 2014. We find that the fraction of insolvent banks is reduced by more than 50%, i.e., from 23% to 10%, which suggests that this buffer is rather efficient to reduce the systemic impact of a bank.

The remainder of the paper is organized as follows: Section 2 presents the model. Section 3 studies the theoretical properties of the equilibrium while Section 4 illustrate our results on the US banking market. Section 5 concludes.

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<sup>3</sup>It is interesting to note that contrary to the applied literature on game theory in IO, see for instance [Bajari et al., 2013], we make here no use of econometric methods. We use data contained in the annual reports of banks, together with our model, to imply the relevant parameters. It is similar in the spirit to the way an implied volatility is computed using the Black Scholes model together with observed price of the vanilla option.

## 2 A model for price-mediated contagion

### 2.1 Banks' balance-sheets and regulatory constraints

Consider a set  $B = \{1, 2, \dots, p\}$  of  $p \geq 2$  banks that can invest in a risky asset and in cash. For each bank  $i$ , we denote by  $v_i > 0$  the amount of cash (in dollars) and by  $q_i P_t > 0$  the value (in dollars) of risky assets, where  $q_i$  is the quantity (in shares) of risky assets held by the bank and  $P_t$  is the market price of the risky asset at a given date  $t$ . Let  $D_i$  be the sum of the value of deposits and/or debt securities, typically coupon bonds, that have been issued by bank  $i$ . From limited liability of stockholders, the value of equity (or capital) at time  $t$  thus is given by:

$$E_{i,t} = \max\{A_{i,t} - D_i; 0\} = \max\{v_i + q_i P_t - D_i; 0\} \quad (2)$$

where  $A_{i,t} = v_i + P_t q_i$  defines the total value of the assets of the bank. The balance-sheet of the bank at time  $t$  is as follows.

#### Balance-sheet of bank $i$ at time $t$

Assets	Liabilities
Cash: $v_i$	Debt: $D_i$
Risky assets: $q_i P_t$	Equity: $E_{i,t}$
$A_{i,t}$	$E_{i,t} + D_i$

In practice, banks may invest in various risky securities (subject to market risk, credit risk...) so that the above balance-sheet, composed with a single risky asset, is a simplified one. We shall explain later on how to make the connection between real balance sheets found in the registration reports of banks and our model.

**Assumption 1** *The risky asset is a financial security issued by a non-financial institution whose price is quoted on financial markets.*

This marketable security is typically a stock index such as the S&P 500 or an ETF replicating a stock index. It can also be a stock issued by a non-financial corporation or even a bond issued by a government or a corporation. Contrary to network models initiated by [Eisenberg and Noe, 2001] and [Cifuentes et al., 2005], (see [Glasserman and Young, 2016] for a recent and comprehensive overview), this financial security is *not* a claim issued by a

bank so that the default of a given bank has no direct impact on the rest of the banking system because there is no direct contractual links (e.g., repo, OTC derivatives...) between banks. Direct contagion of failures is thus not possible in our framework. The unique source of contagion is indirect, through prices. This single risky asset setting is relevant from a regulatory stress-testing perspective as it enables regulators to study worst case scenarios where banks' trading books are highly correlated. It is also motivated by the fact that banks tend to adopt similar behaviors and invest in the same risky assets or have positions in risky assets that can be considered as collinear to a common benchmark.

As discussed in the introduction of this paper, banking regulation imposes banks (i.e., insured depository institutions) to hold enough capital as a *percentage* of the RWA. Since cash is a riskless asset, it does not require any capital so that its regulatory weight is equal to zero. However, a risky asset requires some capital as a function of its risk, measured in some sense, and thus has a positive risk weight. Within our model, since there is a single risky asset, the risk-weighted asset of bank  $i$  is simply equal to

$$\text{RWA}_{i,t} = \alpha_i q_i P_t \quad (3)$$

where  $\alpha_i$  is the risk weight of bank  $i$  associated to the risky asset. Note that  $\alpha_i$  may vary across banks because some of them make use of internal models to compute the RWAs.

Let  $\theta_{i,t}$  be the risk-based capital ratio (RBC) as defined in equation (1) for a given bank  $i$  at time  $t$ :

$$\theta_{i,t} := \frac{E_{i,t}}{\text{RWA}_{i,t}} = \frac{A_{i,t} - D_i}{\alpha_i q_i P_t} > 0 \quad (4)$$

Denote by  $\theta_{min}$  the minimum capital ratio imposed by the regulator. Since  $E_{i,t}$  is the total capital, equal to Tier 1 plus Tier 2,  $\theta_{min}$  thus is equal to 8%. For the sake of interest, we shall assume that before the shock (at time  $t$ ), all banks comply with the regulatory constraint:

$$\theta_{i,t} \geq \theta_{min} \text{ for each } i = 1, 2, \dots, p \quad (5)$$

When one inspects the balance-sheet of an universal bank (i.e., retail/investment banking), as already said, it has positions in many risky assets and not only in a single one. Since the total capital and the total risk weighted assets RWA are publicly disclosed in the annual report, it is possible to imply from the consolidated balance-sheet of each bank the weight  $\alpha_i$  that we shall call an *implied aggregate risk weight*. From the observed balance-sheet,  $A_{i,t} - v_i$  is the total value of the assets minus cash, so that it suffices to set  $q_i P_t = (A_{i,t} - v_i)$  to obtain



the total value of the risky asset. Since both  $E_{i,t}$  and  $\theta_{i,t}$  are also disclosed in the annual report of the bank, from equation (4), it thus follows that

$$\alpha_i = \frac{E_{i,t}}{\theta_{i,t}(A_{i,t} - v_i)} \quad (6)$$

For most large international banks, cash is small compared to the total size of assets (typically less than 5%), which means that  $A_{i,t} - v_i \approx A_{i,t}$  so that

$$\alpha_i \approx \frac{RWA_{i,t}}{\theta_{i,t}A_{i,t}} \quad (7)$$

## 2.2 Impact of an exogenous shock on banks' capital ratios

Assume that a shock on the risky asset occurs at date  $t^+$  and denote  $\Delta \in (0, 1)$  the size of the adverse shock in percentage of  $P_t$ . The price of the risky asset at time  $t^+$  thus is equal to

$$P_{t^+} = P_t(1 - \Delta) \quad (8)$$

Since each bank holds the risky asset,  $\Delta$  is a common adverse shock price that could even be interpreted a systemic shock. At time  $t^+$ , right after the shock, the balance-sheet of bank  $i$  is given as follows.

### Balance-sheet at time $t^+$

Assets	Liabilities
Cash: $v_i$	Debt: $D_i$
Risky assets: $q_i P_t(1 - \Delta)$	Equity: $E_{i,t^+}$
$A_{i,t^+}$	$E_{i,t^+} + D_i$

and it is the role of equity to absorb the shock (i.e., the loss). The RBC of bank  $i$  is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{RWA_{i,t^+}} = \frac{\max\{E_{i,t} - q_i P_t \Delta; 0\}}{\alpha_i q_i P_t (1 - \Delta)} \quad (9)$$

and is a decreasing function of the shock size. A given bank  $i$  may thus be in one of the three following situations, depending on the size of the shock  $\Delta$ :

1. solvent and complying with regulatory capital requirement, that is  $\theta_{i,t^+}(\Delta) \geq \theta_{min}$
2. solvent but not complying with regulatory capital requirement, that is  $0 < \theta_{i,t^+}(\Delta) < \theta_{min}$

3. insolvent, that is  $\theta_{i,t^+}(\Delta) = 0$ , which is equivalent to  $E_{i,t} - q_i P_t \Delta \leq 0$

Let us define two important thresholds:

$$\Delta_i^{sale} : \quad \inf\{\Delta \in [0, 1] : \theta_{i,t^+}(\Delta) = \theta_{min}\} \quad (10)$$

$$\Delta_i^{fail} : \quad \inf\{\Delta \in [0, 1] : E_{i,t^+}(\Delta) = 0\} \quad (11)$$

The following lemma characterizes those two thresholds for each bank and follows directly from equations (10) and (11):

**Lemma 1** *Each bank  $i \in B$  is characterized by the two following critical thresholds.*

$$\Delta_i^{sale} : \quad = \frac{E_{i,t} - \alpha_i \theta_{min} q_i P_t}{q_i P_t (1 - \alpha_i \theta_{min})} = \frac{\Delta_i^{fail} - \alpha_i \theta_{min}}{1 - \alpha_i \theta_{min}} > 0 \quad (12)$$

$$\Delta_i^{fail} : \quad = \frac{E_{i,t}}{q_i P_t} > 0 \quad (13)$$

$$\text{with } \Delta_i^{sale} < \Delta_i^{fail}$$

The knowledge of  $\Delta_i^{sale}$  and  $\Delta_i^{fail}$  enables to predict the situation of bank  $i$  after a shock and anticipate potential reactions. If  $\Delta \geq \Delta_i^{fail}$ , the shock will leave bank  $i$  insolvent, while if  $\Delta \leq \Delta_i^{sale}$  bank  $i$ 's equity will not only absorb the shock but also keep the bank's capital ratio above the minimum regulatory threshold. The interesting scenario, that we explore within this paper, occurs when  $\Delta_i^{sale} < \Delta < \Delta_i^{fail}$  for some  $i$ : in this case, the bank is able to absorb the exogenous shock  $\Delta$  but is left with a regulatory capital ratio that is lower than  $\theta_{min}$ .

Since each bank is characterized by the two thresholds  $\Delta_i^{sale}$  and  $\Delta_i^{fail}$ , the banking system is characterized by  $2p$  thresholds, i.e., by  $(\Delta_i^{sale}, \Delta_i^{fail})$ ,  $i = 1, 2, \dots, p$ . Without loss of generality, we shall assume that<sup>4</sup>:

$$\Delta_1^{fail} \leq \Delta_2^{fail} \leq \dots \leq \Delta_p^{fail} \quad (14)$$

We define now the four following thresholds:

$$\underline{\Delta}^{sale} = \inf_{i \in B} \Delta_i^{sale} \quad \overline{\Delta}^{sale} = \sup_{i \in B} \Delta_i^{sale} \quad (15)$$

$$\underline{\Delta}^{fail} = \inf_{i \in B} \Delta_i^{fail} = \Delta_1^{fail} \quad \overline{\Delta}^{fail} = \sup_{i \in B} \Delta_i^{fail} = \Delta_p^{fail} \quad (16)$$

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<sup>4</sup>This order can be obtained through an elementary permutation.

## 2.3 Endogenous fire sales and feedback effects

Since  $\Delta$  is a common shock, it affects the balance-sheet of *all* banks that hold the risky asset and may leave some of them undercapitalized. Banks that do not comply with the regulatory capital constraints may restore their capital ratio above the minimum required  $\theta_{min}$  in two main ways.

1. They may issue new shares and hence increase the numerator of the risk-based capital ratio.
2. They may also sell assets and decrease the denominator of the risk-based capital ratio.

After such a common shock, e.g., what happened during the subprime crisis, it may be difficult for such banks to sell new stocks. In such a situation, the unique solution for a given bank  $i$  to restore its regulatory capital ratio back above the minimum required is to *sell assets*. In line with the existing literature on the subject (e.g., [Brunnermeier and Oehmke, 2014, Cifuentes et al., 2005, Greenlaw et al., 2012, Greenwood et al., 2015]) and consistent with observed behavior of banks, we make the assumption that undercapitalized banks can only engage in asset sales in order to restore their capital ratio. As in most models, e.g., [Elliott et al., 2014, Caccioli et al., 2014], when a bank is unable to restore its capital ratio above  $\theta_{min}$ , we assume that it is fully liquidated at date  $t + 1$ .

In practice, banks are sometimes able to issue stocks despite the situation of distress. To give a recent example, following its problems with the American justice, Deutsche Bank decided in March 2017 to issue 687.5 million stocks for a total value of \$8 Billion, which is approximately equal to the amount of the fine imposed by the US Justice (\$7.2 Billion). It is important to point out at this stage that it wouldn't be difficult to introduce in our model the possibility for each bank to recapitalize up to a certain amount, say (at most) a given fraction of their existing total capital. In such a case, after a large shock, if the recapitalization is not sufficient for the bank to comply with the regulatory capital ratio, the unique possibility for the banks is to sell assets and we are back to our model. Allowing each bank to recapitalize up to a certain proportion of its existing total capital would thus only change the two thresholds  $\Delta_i^{sale}$  and  $\Delta_i^{fail}$ .

We denote by  $x_i \in [0, 1]$  the proportion of risky assets sold by bank  $i$  at date  $t + 1$ , in reaction to the shock  $\Delta$  at date  $t^+$ . When bank  $i$  does not need to liquidate assets, then

$x_i = 0$ . On the contrary, when the shock  $\Delta$  is such that bank  $i$  is insolvent or unable to restore its capital ratio above  $\theta_{min}$ , then it is fully liquidated and  $x_i = 1$ . The volume (in shares) of liquidation by bank  $i$  is denoted by  $x_i q_i$  and  $\sum_{i \in B} x_i q_i$  denotes the total volume of fire sales in the banking system at date  $t + 1$ .

Fire sales obviously impact the price of the asset at date  $t + 1$  and we assume here this price impact to be linear. We introduce the asset market depth  $\Phi$  which is a linear measure of the asset liquidity [Kyle and Obizhaeva, 2016]. In [Cont and Wagalath, 2016], it is shown that the relevant quantity to capture the magnitude of feedback effects is  $\frac{\sum_{i \in B} q_i}{\Phi}$ . The greater this parameter, the greater the size of the banking system compared to asset market depth and the greater the feedback. The asset price at date  $t + 1$  thus depends on the vector of liquidations  $\mathbf{x}(\Delta, \Phi) := \mathbf{x} = (x_1, x_2, \dots, x_p) \in [0, 1]^p$ , and this vector of liquidation depends on both the shock  $\Delta$  and the market depth  $\Phi$ .

**Assumption 2** *The price of the risky asset at time  $t + 1$  is equal to*

$$P_{t+1}(\mathbf{x}, \Phi) = P_t (1 - \Delta) \left( 1 - \frac{\sum_{i \in B} x_i q_i}{\Phi} \right) \quad (17)$$

$$\frac{Q_{tot}}{\Phi} < 1 \quad (18)$$

$$\text{where } Q_{tot} = \sum_{i \in B} q_i \quad (19)$$

In a more general model, one could allow the price impact to be a non linear function of the quantity sold. For instance, it could be possible, as in [Cifuentes et al., 2005], to consider a convex price impact function in which  $P_{t+1}(\mathbf{x}, \Phi) = P_t (1 - \Delta) e^{-\frac{\sum_{i \in B} x_i q_i}{\Phi}}$ . This would only complicate the analysis without adding new economic insight. In any event, as we shall see later on, the proof of the existence of the equilibrium would also work in the non-linear case. The assumption that  $\frac{Q_{tot}}{\Phi} < 1$  is aimed at keeping the price strictly positive even if all banks fully liquidate their positions on the risky asset.

As discussed above, the market depth  $\Phi$  is a linear measure of the risky asset's liquidity: the larger this parameter, the more liquid this asset. When  $\Phi = \infty$ , the asset is infinitely liquid and fire sales do not impact the asset price. In practice, roughly speaking, the asset liquidity is measured by the presence of buyers *outside* the banking system when banks need to sell the risky assets. Such potential buyers may typically be hedge funds that can absorb fire sales from banks and limit their impact on asset prices. As such, our model shows the

(relative) importance of the shadow banking as a possible stabilizing force in the case of a banking crisis. Note however that hedge funds might also face funding difficulties during a banking crisis, which may limit their ability to buy back assets from distressed banks [Caballero and Simsek, 2013]. In our model, this kind of difficulty would be translated into a lower  $\Phi$ . In a dynamic model,  $\Phi$  could be time-dependent and an evaporation of the liquidity would be modeled by a sharp fall of  $\Phi$ .

At time  $t + 1$ , the balance-sheet of bank  $i$  that sold a portion  $x_i$  of the risky asset is given below:

**Balance-sheet of bank  $i$  at date  $t + 1$  after deleveraging**

Assets	Liabilities
Cash: $v_i + x_i q_i P_{t+1}(\mathbf{x}, \Phi)$	Debt: $D_i$
Risky asset: $(1 - x_i) q_i P_{t+1}(\mathbf{x}, \Phi)$	Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + q_i P_{t+1}(\mathbf{x}, \Phi)$	$E_{i,t+1} + D_i$

where  $P_{t+1}(\mathbf{x}, \Phi)$  is given in Assumption (2). Let  $E_{i,t+1}(\mathbf{x})$  be the total capital at time  $t + 1$  after deleveraging. From the above balance-sheet, it is not difficult to show that

$$E_{i,t+1}(\mathbf{x}, \Delta) = \max \left\{ E_{i,t} - q_i P_t \left( \Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi} (1 - \Delta) \right); 0 \right\} \quad (20)$$

and note that it is a *decreasing* function of  $x_i$  due to the existence of a price impact. The regulatory capital ratio of bank  $i$  at time  $t + 1$  (i.e., after deleveraging) thus is equal to

$$\theta_{i,t+1}(\mathbf{x}, \Delta) = \frac{E_{i,t+1}(\mathbf{x})}{\alpha_i q_i P_{t+1}(\mathbf{x}, \Phi) (1 - x_i)} \quad (21)$$

with the natural convention that  $\theta_{i,t+1}(\mathbf{x}, \Delta) = 0$  when  $x_i = 1$  and when  $E_{i,t+1} = 0$ . Let us now introduce the concept of the *implied shock*:

$$\Delta(\mathbf{x}) := \Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi} (1 - \Delta) \quad (22)$$

associated to the vector of liquidation  $\mathbf{x}$  such that the price of the risky asset at date  $t + 1$  can be written as follows

$$P_{t+1}(\mathbf{x}, \Phi) = P_t (1 - \Delta(\mathbf{x})) \quad (23)$$

As long as  $\mathbf{x} \neq \mathbf{0}$ ,  $\Delta(\mathbf{x}) > \Delta$  so that fire sales at date  $t + 1$  actually reinforce the underperformance of the asset caused by the initial shock  $\Delta$  at date  $t^+$ . By re-inserting the

implied shock in equation (21) and by dividing the numerator and the denominator by  $q_i P_t$ , equation (21) reduces to

$$\theta_{i,t+1}(\mathbf{x}, \Delta) = \frac{\max\{\Delta_i^{fail} - \Delta(\mathbf{x}); 0\}}{\alpha_i(1 - x_i)(1 - \Delta(\mathbf{x}))} \quad (24)$$

so that we immediately obtain the following equivalence.

$$E_{i,t+1}(\mathbf{x}, \Delta) > 0 \iff \Delta_i^{fail} - \Delta(\mathbf{x}) > 0 \quad (25)$$

**Assumption 3** *Each bank  $i = 1, 2, \dots, p$  rebalances its portfolio of assets in order to minimize  $x_i \in [0, 1]$  subject to the constraint*

$$\theta_{i,t+1}(\mathbf{x}, \Delta) \geq \theta_{min} \quad (26)$$

*If the constraint can not be satisfied for some  $x_i \in [0, 1]$ , then, bank  $i$  is insolvent and is costlessly liquidated at time  $t + 1$  so that  $x_i = 1$ .*

In [Caccioli et al., 2014], [Cont and Schaanning, 2016] and [Greenwood et al., 2015], they also consider a liquidation problem with price impact but in which each bank is assumed to make use of a purely mechanistic decision rule to liquidate its assets (e.g., the proportional liquidation rule). As a result, as observed in [Caccioli et al., 2014], the sequence of liquidation over time becomes closely related to contagion processes in epidemiology. In this paper, we follow a different approach since we explicitly recognize the strategic feature of the liquidation problem. From equation (21), the RBC of a given bank  $i$  is influenced by the decisions of *all* banks so that a given bank can not decide *independently* of the other banks the minimum portion of the risky asset to liquidate. The problem is similar to a Cournot oligopoly but somehow more complex as each bank is explicitly faced with a regulatory constraint. Since all liquidation decisions are taken at time  $t + 1$  only, we look for a static Nash equilibrium.

When a corporation such as bank (or even a non financial institution) is liquidated, one can identify two main types of costs associated with the failure of that institution [Glasserman and Young, 2016]. Administrative and legal costs called bankruptcy cost on the one hand and costs of delay in making payments on the other hand. In (banking) network models, everything else equal, the greater the bankruptcy costs when bank  $i$  fails, the lower the recovery rate of its claimants (e.g., the other banks in the banking system). As

a result, bankruptcy costs increase the magnitude and the likelihood of failure cascades<sup>5</sup>. Within our model, as there is no direct links between banks, bankruptcy costs *do not affect* the likelihood nor the magnitude of the contagion process. Since our aim is to focus on failure contagion, without loss of generality, we assume that these bankruptcy costs are equal to zero. Of course, if our aim was to analyze the cost borne say by depositors (or bondholders) in case of failure of banks, then, bankruptcy costs would play an important role. Note interestingly that in Europe, it is actually the role of the recent single resolution mechanism, the second pillar of the banking union “to ensure the efficient resolution of failing banks with minimal costs for taxpayers and to the real economy<sup>6</sup>”. It is also stated that the resolution of a bank could be done over a week-end...

Let  $(D_i, q_i, v_i)$  be the triplet that characterizes a given bank  $i$  in our model. When a bank  $j$  knows the triplet that characterizes bank  $i$ , since the current market price is observable, bank  $j$  is able to determine  $A_i, E_i, \alpha_i$ , the RBC and the two thresholds  $\Delta_i^{sale}$  and  $\Delta_i^{fail}$ . Let  $\mathcal{S} = \{(D_i, q_i, v_i)_{i=1}^p; \Phi\}$  be the *structure of the game*.

**Assumption 4** *Information is complete, i.e., the structure of the game  $\mathcal{S}$  as well as the decision-rule of each bank is common knowledge.*

Within our framework, since each bank is assumed to be exposed to a security issued by non-financial institution, as already discussed, there is no network of interconnections between banks. As a result, a given bank only needs to know the balance sheet of the other banks and the price impact to determine whether or not it may be indirectly impacted by fire sales. Given the various information that are publicly disclosed in the annual reports of banks, complete information is justified.

### 3 Strategic fire sales: Nash equilibrium with strategic complementarities

The timing of the *liquidation game* is by assumption as follows.

1. Right after the shock, the price of the risky asset decreases by a percentage  $\Delta > 0$ .

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<sup>5</sup>See once again the comprehensive review of [Glasserman and Young, 2016] for the literature on the subject.

<sup>6</sup>See <https://www.bankingsupervision.europa.eu/about/bankingunion/html/index.en.html>

2. At time  $t + 1$ , each bank  $i = 1, \dots, p$  liquidates a portion  $x_i \in [0, 1]$  of the risky asset, so that the implied shock is equal to  $\Delta(\mathbf{x}) = \Delta + (1 - \Delta) \left( \frac{\sum_{i \in B} x_i q_i}{\Phi} \right)$ .

### 3.1 Best responses and upward jumps

As usual in game theory, let  $\mathbf{x} = (x_i, x_{-i})$  where  $x_{-i} \in [0, 1]^{p-1}$  is a  $p - 1$ -dimensional vector and let us write the vector of liquidation as  $\mathbf{x} = (BR_i(x_{-i}), x_{-i})$ , where  $BR_i(x_{-i})$  is the *unique best response* of bank  $i$  given  $x_{-i} \in [0, 1]^{p-1}$  in the sense of the minimization problem given in assumption 3. Within our model, as there is no direct link between banks, each bank is impacted by the decision of all the other banks only through the price impact. Let

$$\mathcal{S}_i^v = \{x_{-i} \in [0, 1]^{p-1} : \sum_{j \neq i} q_j x_j = v\} \quad (27)$$

From equation (22), since the implied shock depends on  $x_{-i} \in [0, 1]^{p-1}$  only through the sum of its components  $\sum_{j \neq i} q_j x_j$ , for bank  $i$ , each  $x_{-i} \in \mathcal{S}_i^v$  yields the *same* unique best response. As a result, for any shock  $\Delta > 0$ , as long as  $x_{-i} \in \mathcal{S}_i^v$ , the unique best response of bank  $i$  can be written as a function of  $v$ , i.e.,  $x_i^* = BR_i(v, \Delta)$ . Assume that  $x_{-i} \in \mathcal{S}_i^v$  for some  $v > 0$ . The implied shock defined in (22) thus can be written as a function of two variables  $x_i$  and  $v$ .

$$\Delta(x_i, v) = \Delta + (1 - \Delta) \left( \frac{x_i q_i + v}{\Phi} \right) \quad (28)$$

and is an *increasing function* of  $v$  and of  $x_i$ . Let  $BR_i(v, \Delta)$  be the best response of bank  $i$  and note that this best response also depends on  $\Phi$ .

**Lemma 2** 1. For a given  $\Delta$  and for each  $i = 1, \dots, p$ , if  $v_2 \geq v_1$ , then,  $BR_i(v_2, \Delta) \geq BR_i(v_1, \Delta)$ .

2. Let  $v > 0$ . For each  $i = 1, \dots, p$ , if  $\Delta_2 \geq \Delta_1$ , then,  $BR_i(v, \Delta_2) \geq BR_i(v, \Delta_1)$ .

**Proof.** See the appendix.

Part 2 of the above lemma says, as one can expect, that each bank needs to sell more risky assets when the shock  $\Delta$  is larger, everything else equal. In the same vein, part 1 states that the best response of a given bank  $i$  increases with  $v$ , which means that bank  $i$  has an incentive to liquidate more risky assets when the other banks increase their volume of fire sales. In economic theory, this property of monotone increasing best response is called



*strategic complementarity* [Vives, 1990]. We shall now show that the best response needs not be a continuous function of  $v$ .

**Lemma 3**  $BR_i(v, \Delta)$  needs not be a continuous function of  $v$ , it may exhibit an upward jump.

We believe that it is important to understand why such a discontinuity may occur within our framework. As a consequence, we present the argument of the proof in the text. Assume that  $\Delta \in (\Delta_i^{sale}, \Delta_i^{fail})$  and denote  $\theta_{i,t+1}(x_i, v, \Delta)$  the capital ratio of bank  $i$  after the deleveraging process. It is cumbersome but not difficult to show that

$$\frac{\partial \theta_{i,t+1}}{\partial x_i}(0, v, \Delta) > 0 \iff (\Delta_i^{fail} - \Delta(0, v))(1 - \Delta(0, v)) > \frac{q_i}{\Phi}(1 - \Delta)(1 - \Delta_i^{fail}) \quad (29)$$

which leads to a quadratic equation in  $v$  because of the term  $\Delta^2(0, v)$ . When  $\Delta_i^{fail} - \Delta(0, v) > 0$ , the above inequality will be satisfied when  $\Phi$  is large enough. Assume moreover that there exists  $\tilde{x}_i < 1$  such that  $\Delta_i^{fail} - \Delta(\tilde{x}_i, v) = 0$ , that is, the total capital as well as the capital ratio of bank  $i$  are equal to zero for  $x_i \in (\tilde{x}_i, 1)$ . Since the numerator of the RBC is a *linear* function of  $x_i$  while its denominator is a *quadratic* function of  $x_i$ , there is a unique maximum  $x_i^0 \in (0, \tilde{x}_i)$  such that  $\frac{\partial \theta_{i,t+1}}{\partial x_i}(x_i^0, v, \Delta) = 0$ , i.e., the RBC is a *single-peaked* function of  $x_i$ . If  $\theta_{i,t+1}(x_i^0, v, \Delta) > \theta_{min}$ , it is clear that  $x_i^* := BR_i(v, \Delta) < x_i^0$ . Consider now  $v' > v$  and assume that the RBC is still a single peaked function such that  $\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, v', \Delta) < \theta_{min}$ . Bank  $i$  must thus fail so that  $BR_i(v', \Delta) = 1$ . From the discussion, there exists  $v_d \in (v, v')$  such that  $\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, v_d, \Delta) = \theta_{min}$  (see Fig 1) so that  $BR_i(v_d, \Delta) < 1$ . However, for all  $\epsilon > 0$ ,  $BR_i(v_d + \epsilon, \Delta) = 1$  so that the best response is not *right continuous* in  $v_d$ . It is indeed neither right continuous in  $\Delta$ . The best response exhibits an upward jump at  $v_d$  and the size of this jump is equal to  $1 - BR_i(v_d, \Delta) > 0$ .

In the classical proof of the existence of a Nash equilibrium, the application of a fixed point theorem such as the Brouwer's one requires the best response to be continuous. However, as noticed in their early paper, [Roberts and Sonnenschein, 1976] proved the existence of the Nash equilibrium assuming that the discontinuities take the form of upward jumps, see [Vives, 1990] section 7 or [Milgrom and Roberts, 1994] p. 447 for a discussion. Within our framework, depending on the parameters, as we have seen, one can not exclude such upward jumps, which means that a more powerful result than the classical Brouwer's fixed point theorem should be used.

### Existence of an upward jump

$\varepsilon > 0$  is arbitrarily small

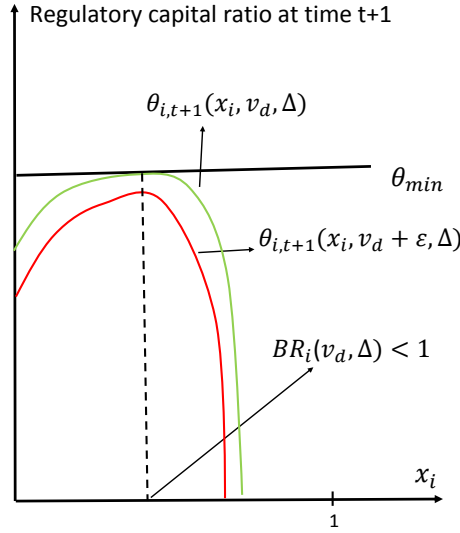


Figure 1: The best response may exhibit a discontinuity

## 3.2 Existence of a Nash equilibrium

We first start by giving the definition of a Nash equilibrium in our model.

**Definition 1** For a given initial shock  $\Delta > 0$ , the vector of liquidation  $\mathbf{x}^* = (x_1^*, \dots, x_p^*) \in [0, 1]^p$  is a Nash equilibrium if and only if for all  $i = 1, 2, \dots, p$ :

$$BR_i(x_{-i}^*, \Delta) := x_i^* = \min \{x_i \in [0, 1) \text{ such that } \theta_{i,t+1}(x_i, x_{-i}^*, \Delta) \geq \theta_{min}\} \text{ or } x_i^* = 1 \quad (30)$$

Saying that  $\mathbf{x}^*$  is a Nash equilibrium means that, for each  $i = 1, 2, \dots, p$ , the best response  $BR_i(x_{-i}^*, \Delta)$  is equal to  $x_i^*$ . At equilibrium, the implied shock  $\Delta(\mathbf{x}^*) \equiv \Delta^*$  thus is equal to

$$\Delta^* = \Delta + \left( \frac{\sum_{i \in B} x_i^* q_i}{\Phi} \right) (1 - \Delta) \quad (31)$$

To prove the existence of a Nash equilibrium within our framework, we shall use a fixed point result that requires some basic preliminaries. Recall that  $X = [0, 1]^p$  is the set of all liquidation vectors and consider now the pair  $(X, \geq)$  where  $\mathbf{x} \leq \mathbf{y} \iff x_i \leq y_i$  for each  $i = 1, \dots, p$  so that  $(X, \geq)$  is a partially ordered set (poset for short). A poset  $(X, \geq)$  is

said to be a lattice if, for any pair of elements  $\mathbf{x}$  and  $\mathbf{y}$  of  $X$ , the supremum  $\sup\{\mathbf{x}, \mathbf{y}\}$  and the infimum  $\inf\{\mathbf{x}, \mathbf{y}\}$  exist in  $X$ . The lattice is said to be complete if, for all non-empty subset  $E \subset X$ , the supremum  $\sup E$  and the infimum  $\inf E$  exist in  $X$ . See for instance [Tarski, 1955],[Vives, 1990], [Milgrom and Roberts, 1990]. When  $X$  is the product of  $p$  compact sets, i.e.,  $X = [0, 1]^p$ , it is well-known, and easy to see, that the poset is a complete lattice. The following result is due to [Tarski, 1955] and, to the best of our knowledge, has been introduced in economic theory by [Vives, 1990] and [Milgrom and Roberts, 1990]. It has been subsequently used in finance by [Eisenberg and Noe, 2001] in their their influential paper on systemic risk.

**Tarski's theorem** ([Tarski, 1955], see also [Vives, 1990] or [Milgrom and Roberts, 1990]). *Let  $(L, \geq)$  be a complete lattice and  $f$  a non decreasing function from  $L$  to  $L$  and  $\mathcal{F}$  the set of fixed points of  $f$ . Then,  $\mathcal{F}$  is non-empty and  $(\mathcal{F}, \geq)$  is a complete lattice. In particular,  $\sup_{\mathbf{x}} \mathcal{F}$  and  $\inf_{\mathbf{x}} \mathcal{F}$  belong to  $\mathcal{F}$ .*

As observed by [Vives, 2001] in his well-known textbook on oligopoly pricing, this fixed point result is interesting as it does not make any use of topological properties such as compactness or continuity. It only requires the function  $f$  to be non-decreasing, which is the case in our framework. The linearity of the price impact function thus plays no role in the proof of the proposition below and a non-linear price impact function would not change the existence result.

**Proposition 1** *For all initial shock  $\Delta \in (0, 1)$  and market depth  $\Phi > 0$ , the set of Nash equilibrium denoted  $\mathcal{F}_\Delta$  is not empty. For any equilibrium  $\mathbf{x}^* \in \mathcal{F}_\Delta$ , the subset of banks that are solvent and insolvent, denoted  $S^*$  and  $D^*$  respectively, when non empty, are composed with consecutive integers, i.e., there exists  $0 \leq i(\mathbf{x}^*) := i^* \leq p$  such that  $D^* = \{1, \dots, i^*\}$  and  $S^* = \{i^* + 1, \dots, p\}$ .*

**Proof.** See the appendix.

Note importantly that throughout this paper, when  $\mathcal{F}_\Delta$  contains more than one Nash equilibrium, we shall always consider the smallest one, that is, the one that minimizes the implied shock  $\Delta(\mathbf{x}^*)$ , or equivalently the total amount liquidated equal to the scalar product  $\mathbf{x}^* \mathbf{q}$ .

It is well-known that games with strategic complementarities can have more than one Nash equilibrium which may be Pareto ranked (e.g., [Vives, 2005]) and our model is of no exception. From a financial point of view, we claim that it makes only sense to consider the *smallest* Nash equilibrium. Assume for the discussion that  $\mathcal{F}_\Delta = \{\mathbf{x}^*, \mathbf{y}^*\}$ . Since these two equilibria are ordered, say  $\mathbf{y}^* \geq \mathbf{x}^*$ , it is in the interest of *all market participants* – banks, depositors, bondholders – to choose the Nash equilibrium that minimizes the market impact since it also minimizes the number of insolvent banks. As noted by [Fudenberg and Tirole, 1991] in their well-known textbook on game theory, choosing a particular equilibrium relies on some mechanism that leads all the banks to expect the same equilibrium, which thus becomes the focal point. In general, the explanation of this choice is based on *preplay communication*, that is, on the possibility for the banks to "talk" before the game. As usual in the literature, when Nash equilibria are Pareto ranked, we make here the assumption that banks are able to coordinate on the Pareto-dominant equilibrium<sup>7</sup>, that is, on the smallest Nash equilibrium within our model. Since supervisors monitor financial stability and are reluctant to let large institutions fail, the smallest Nash equilibrium is clearly also the preferred solution of regulatory public institutions. In such a case, since each bank expects the rest of the banking sector to choose the smallest one, it is naturally a focal point.

### 3.3 Characterization of Nash equilibrium and convexity effects

When there is no price impact, i.e.,  $\frac{1}{\Phi} = 0$ , the liquidation problem turns out to be very simple since the problem is not anymore strategic. In such a situation, when  $\Delta \in (\Delta_i^{sale}, \Delta_i^{fail})$ , after deleveraging, the RBC of bank  $i$  is equal to

$$\theta_{i,t+1}(x_i) = \frac{E_{i,t} - q_i \Delta P_t}{\alpha_i q_i P_t (1 - \Delta)(1 - x_i)} \quad (32)$$

and the numerator of equation (32) is *invariant* with respect to  $x_i$ . Since the denominator is an decreasing function of  $x_i$  that converges to zero, there exists a unique solution  $\underline{x}_i^* < 1$  such that  $\theta_{i,t+1}(\underline{x}_i^*) = \theta_{min}$ . The solution is equal to

$$\underline{x}_i^* = 1 - \left[ \frac{\Delta_i^{fail} - \Delta}{\alpha_i (1 - \Delta) \theta_{min}} \right] < 1 \quad (33)$$

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<sup>7</sup>In [Fudenberg and Tirole, 1991] paragraph 1.2.4 entitled *Multiple Nash equilibria, Focal points and Pareto Optimality*, they offer an example of a particular game in which players may not choose the Pareto dominant equilibrium but rather the risk dominant one, which is Pareto dominated.

Thus, the optimal portion of the risky asset to liquidate for each bank  $i = 1, \dots, p$  as a function of  $\Delta$  can be written as follows.

- If  $\Delta \leq \Delta_i^{sale}$ , then  $\underline{x}_i^* = 0$
- If  $\Delta_i^{sale} < \Delta < \Delta_i^{fail}$ , then

$$\underline{x}_i^* = 1 - \left( \frac{1 - \Delta_i^{sale}}{1 - \Delta} \right) \left( \frac{\Delta_i^{fail} - \Delta}{\Delta_i^{fail} - \Delta_i^{sale}} \right) \quad (34)$$

- If  $\Delta \geq \Delta_i^{fail}$  then  $\underline{x}_i^* = 1$

Denote  $x_+ := \max\{x; 0\}$  and recall that the payoff of a call option with strike price  $K$  when the underlying asset is equal to  $x$  is equal to  $(x - K)_+$ . For a given bank  $i$ , the amount liquidated (in dollars) is equal to  $\underline{x}_i^* q_i P_t (1 - \Delta)$  and, given equation (34), the amount liquidated is equal to:

$$q_i P_t \left( \frac{1 - \Delta_i^{fail}}{\Delta_i^{fail} - \Delta_i^{sale}} \right) \underbrace{\left[ (\Delta - \Delta_i^{sale})_+ - (\Delta - \Delta_i^{fail})_+ \right]}_{\text{Difference between two call options}} - q_i P_t (\Delta - \Delta_i^{fail})_+ \quad (35)$$

This amount liquidated can actually be expressed as the *difference between two call options*, where  $\Delta$  plays the role of the underlying asset and the thresholds  $\Delta_i^{sale}$  and  $\Delta_i^{fail}$  the role of the strike prices. Since the payoff of call options are convex in the underlying assets, i.e., here in  $\Delta$ , equation (35) shows that the amount liquidated is indeed the difference of two convex functions. This means that the amount liquidated is (highly) convex when  $\Delta$  is close to  $\Delta_i^{sale}$ . In the next section, we shall actually provide empirical examples of this convexity effect.

We are now in a position to characterize the Nash equilibrium in the case of a *small shock*, when all banks sell a portion of the risky asset and survive the deleveraging process, that is, when for all  $1 \leq i \leq p$ :

$$\Delta_i^{fail} > \Delta(\mathbf{1}) \iff E_i - q_i P_t \Delta - \frac{Q_{tot}}{\Phi} q_i P_t (1 - \Delta) > 0 \quad (36)$$

where  $\mathbf{1} := (1, \dots, 1)$ . The above inequality is equivalent to:

$$\Delta < \frac{\Delta_i^{fail} - \frac{Q_{tot}}{\Phi}}{1 - \frac{Q_{tot}}{\Phi}} \quad (37)$$

and note that the rhs of equation (37) is positive for bank 1 (and thus for all banks) when  $\Phi$  high enough. Recall that  $x_i^*$  is the optimal portion of risky asset liquidated by bank  $i$  (at equilibrium) when there is a positive price impact, i.e., when  $\frac{1}{\Phi} > 0$ . It should be clear that for each  $i = 1, \dots, p$ ,  $x_i^* \geq \underline{x}_i^*$ . Let  $Q^* = \sum_{i=1}^p x_i^* q_i$  be the total amount liquidated when the price impact is positive and let  $\underline{Q}^* = \sum_{i=1}^p \underline{x}_i^* q_i$  be this total amount liquidated when there is no price impact, where  $\underline{x}_i^*$  is given by equation (33). Since  $x_i^* \geq \underline{x}_i^*$  as long as the price impact is positive, the characterization of  $\Delta x_i^* := x_i^* - \underline{x}_i^*$  would be a valuable result to (better) understand the sensitivity of  $\Delta x_i^*$  as a function of the parameters of our model. It is precisely the aim of the following proposition to provide such a characterization. Recall that  $\underline{\Delta}^{fail}$  and  $\overline{\Delta}^{sale}$  are defined respectively in equations (15) and (16). In the following proposition, given the shock, it is assumed that each bank needs to sell a positive portion of the risky asset but also that each bank survived the deleveraging process (at equilibrium). As a result, at time  $t + 1$ ,  $\theta_{i,t+1}(\cdot) = \theta_{min}$  for all  $1 \leq i \leq p$  and this property allows us to make further computation.

**Proposition 2** *Assume that the initial shock  $\Delta > 0$  is such that:*

$$\overline{\Delta}^{sale} < \Delta < \frac{\underline{\Delta}^{fail} - \frac{Q_{tot}}{\Phi}}{1 - \frac{Q_{tot}}{\Phi}} \quad (38)$$

*Then*

$$x_i^* = \underline{x}_i^* + \left( \frac{Q^*}{\Phi} \frac{1 - \Delta_i^{fail}}{\theta_{min} \alpha_i (1 - \Delta)} \right) + o\left(\frac{1}{\Phi}\right) \quad (39)$$

*so that the total quantity sold is equal to*

$$Q^* = \underline{Q}^* \times \left( 1 + \frac{1}{\Phi} \sum_{i=1}^p \frac{q_i (1 - \Delta_i^{fail})}{\theta_{min} \alpha_i (1 - \Delta)} \right) + o\left(\frac{1}{\Phi}\right) \quad (40)$$

**Proof.** See the appendix.

In the case of a small shock, the above proposition shows that when all the banks remain solvent after the deleveraging process,  $x_i^*$  can be expressed as the sum of  $\underline{x}_i^*$ , the optimal portion to liquidated when there is no price impact, and a positive quantity which depends on the parameters of the model only. From equation (39), everything else equal, when the size of the shock  $\Delta$  increases, the optimal portion of the risky asset liquidated by each bank  $i$  increases while when  $\theta_{min}$  (or  $\alpha_i$ ) increases, it decreases. Finally, as expected,  $x_i^*$  tends to  $\underline{x}_i^*$  in the limiting case in which  $\Phi$  tends to infinity.

### 3.4 Price mediated-contagion and amplification effect of a marginal shock

Within our model, due to price impact, the liquidation process of a given bank may adversely impacts the capital ratio other banks, which may ultimately leads to the failure of some of them through price-mediated contagion. Because of the heterogeneity in banks' capital structures and regulatory weights, captured by the distribution of thresholds  $(\Delta_i^{sale}, \Delta_i^{fail})_{1 \leq i \leq p}$ , the fraction of insolvent banks is a non-linear function of the shock size even when there is no price impact. To correctly measure the amplification effect associated to a marginal shock, we thus consider the no price impact case as a benchmark. Assume for instance that the marginal shock is equal to 100 bps. If the resulting marginal fraction of insolvent banks is equal to 5% when there is no price impact, one can only say that there is an amplification effect in the positive price impact case if the marginal fraction of insolvent banks is *greater than* 5%.

Assume that the initial shock is such that  $\Delta = \Delta_j^{fail}$  for some  $j \in \{1, 2, \dots, p-1\}$  and denote  $\lambda(\Delta^*(\Delta, \frac{1}{\Phi})) := \lambda^*(\Delta, \frac{1}{\Phi})$  the fraction of insolvent banks *at equilibrium*. When there is no price impact, that is  $\frac{1}{\Phi} = 0$ , the subset of banks that are insolvent is equal to  $\{1, \dots, j\}$  so that  $\lambda^*(\Delta, 0) = \frac{j}{p}$ . When the initial shock increases by  $\delta \geq 0$ , that is, the shock is now equal to  $\Delta' := \Delta + \delta$ , the fraction of insolvent banks increases and is now equal to  $\lambda^*(\Delta + \delta, 0) = \frac{m}{p}$ , for some  $m \in \{j, \dots, p-1\}$ . The marginal fraction of insolvent banks due to the marginal shock  $\delta$  is positive and equal to  $\lambda^*(\Delta + \delta, 0) - \lambda^*(\Delta, 0) = \frac{m-j}{p} \geq 0$ . Of course, this quantity is positive not because of price-mediated contagion due to the existence of a positive price impact, but simply because a larger shock, depending on the distribution of the threshold, triggers additional failures. To measure the possible amplification of the marginal fraction of insolvent banks due to the existence of a positive price impact, it is thus natural to consider the following ratio

$$\mathcal{I} \left( \Delta, \delta, \frac{1}{\Phi} \right) = \frac{\lambda^*(\Delta + \delta, \frac{1}{\Phi}) - \lambda^*(\Delta, \frac{1}{\Phi})}{\lambda^*(\Delta + \delta, 0) - \lambda^*(\Delta, 0)} \geq 0 \quad (41)$$

which is indeed a ratio of two slopes. We shall say that there is an *amplification effect* at  $(\Delta, \delta, \frac{1}{\Phi})$  if  $\mathcal{I}(\Delta, \delta, \frac{1}{\Phi}) > 1$ . Let  $\delta = 100$  basis points and assume that  $\mathcal{I}(\cdot) = 2$ . This means that the marginal fraction of insolvent banks when there is a price impact is *doubled* compared to the case without price impact<sup>8</sup>. Note that when this measure of amplification

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<sup>8</sup>In the same way, it would be possible to define a similar indicator when the market impact increases for

is greater than one, it may be thought of as a particular measure of *financial fragility* (of a financial system) as defined in [Allen and Gale, 2004].

Note that it may of course be the case that  $\mathcal{I}(\cdot) < 1$ , and indeed equal to zero. To see this, let  $\Delta_{Syst}(\frac{1}{\Phi})$  be the *smallest shock*  $\Delta \leq \overline{\Delta}^{fail}$  such that all the banks are insolvent at equilibrium, i.e.,  $x_i^* = 1$  for each  $i = 1, \dots, p$ . Assume that  $\Delta_{Syst} < \overline{\Delta}^{fail}$  and that both  $\Delta$  and  $\Delta + \delta$  belong to  $(\Delta_{Syst}, \overline{\Delta}^{fail})$ . It thus follows that  $\lambda^*(\Delta, \frac{1}{\Phi}) = \lambda^*(\Delta + \delta, \frac{1}{\Phi}) = 1$  so that  $\mathcal{I}(\cdot) = 0$ . When the initial shock is large enough and/or the price impact is important, most of the banks are already insolvent for the initial shock so that one naturally expects no amplification effect in such a case.

## 4 Empirical analysis based on the 2015 FED stress tests

Stress testing is the analysis of how a generic entity (or object) such as a human body, a car, a bank... or a system of interacting entities such as a physical, biological or a financial system copes under pressure. For the specific case of a banking system, composed with interacting banks, there are actually various ways to design a stress test to assess its resilience. The stress test can be done on a micro prudential basis (bank by bank) or on a macro prudential basis, on a forward looking basis (based on projections of revenues, losses, capital, RWA in a given scenario) or on a point in time basis, on a specific asset class of assets (banking book or trading book) or on all the asset classes etc... While supervisory stress tests were first coordinated in U.S. right after the events of 2008, as recalled in [Dent et al., 2016], internal stress tests were conducted by banks themselves for risk management purpose in the early 1990 in order for the bank to (better) assess their trading portfolio's losses. This practice of stress tests was actually formalized in 1996 in line with the market risk amendment to Basel accords (see for instance [Dimson and Marsh, 1997]). Since the last few years, American but also European banks are now required to follow the guidelines of supervisors to conduct their stress tests. These *supervisory stress tests* (see [Hirtle and Lehnert, 2015] for a recent overview) are now described and sometimes criticized in various recent academic papers such as [Acharya et al., 2014], [Borio et al., 2014], [Flannery et al., 2017], [Greenlaw et al., 2012].

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a constant shock.



## 4.1 Comprehensive Capital Analysis and Review (CCAR) 2015

In their public document entitled CCAR 2015 summary instructions and guidance<sup>9</sup>, the Board of Governors of the Federal Reserve System reports that the annual CCAR is “an intensive assessment of the capital adequacy of large, complex U.S. bank holding companies (BHC) and of the practices these BHC use to assess their capital needs”. Each bank with consolidated assets of \$50 billion or more is required to participate and there is a total of 31 banks participating. As expected, one finds the well-known active international banks (identified as GSIBs) such as JP Morgan, Citigroup or Bank of America with consolidated assets higher than \$1800 billion. But one also finds smaller and less known banks such as Comerica Incorporated, Discover Financial Services or Zions Bancorporation with consolidated assets lower than \$100 billion. Overall, the document provides the general instructions, qualitative and quantitative, required to perform the stress test. For instance, in section 3 entitled *Stress tests conducted by BHCs*, the document reports that each bank must conduct its stress test using five scenarios, three supervisory scenarios ranked by their severity (baseline, adverse, severely adverse) and two BHC-defined scenarios also ranked by their severity (baseline, stress). One also learns from the document that the stress test is forward looking, from fourth quarter of 2014 to fourth quarter of 2016 so that each BHC is required, as of September 30, 2014, to estimate quantities such as revenues, losses, capital. Few BHCs (the six largest), with large trading operations, will “be required to include a global market shock as part of their supervisory adverse and severely adverse scenarios”. The interesting feature of this global market shock is that it is an exogenous loss in the trading book that may reduce the (total) capital of the BHC. The documents precises that this global market shock is as of October 6, 2014. It further reports that the BHC is not assumed to decrease its portfolio positions or RWAs due to losses from that global shock and this is the point that our approach disputes. Within our framework, we make the assumption, as in the regulatory stress tests, that there is a point in time shock in the trading book. However, we explicitly allow the bank to react. Right after the shock, within our model, the unique possibility for the bank to react is to sell a portion of its assets and we quantify the resulted price-mediated contagion.

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<sup>9</sup>See [Federal-Reserve, 2015].

**Implied risk weight and critical thresholds**

Bank	$\alpha$	$\Delta^{sale}$	$\Delta^{fail}$
Ally Financial Inc	0.8612	0.0485	0.1141
American Express Company	0.8378	0.0683	0.1307
Bank of America Corporation	0.5997	0.0303	0.0768
BB&T Corporation	0.7690	0.0564	0.1144
BBVA Compass Bancshares, Inc	0.7747	0.0398	0.0993
BMO Financial Corp	0.3787	0.0247	0.0542
Capital One Financial Corporation	0.7710	0.0584	0.1164
Citigroup Inc	0.7017	0.0357	0.0898
Citizens Financial Group Inc	0.7976	0.0668	0.1263
Comerica Incorporated	0.9867	0.0268	0.1036
Discover Financial Services	0.8751	0.0854	0.1494
Fifth Third Bancorp	0.8498	0.0577	0.1218
HSBC North America Holdings Inc	0.4631	0.0367	0.0724
Huntington Bancshares Incorporated	0.8217	0.0489	0.1114
JPMorgan Chase &Co	0.6295	0.0315	0.0803
KeyCorp	0.9070	0.0576	0.1260
M&T Bank Corporation	0.8002	0.0616	0.1217
Morgan Stanley	0.5689	0.0503	0.0935
MUFG Americas Holdings Corporation	0.8507	0.0615	0.1254
Northern Trust Corporation	0.5721	0.0421	0.0859
Regions Financial Corporation	0.8278	0.0641	0.1260
Santander Holdings USA, Inc	0.5635	0.0401	0.0833
State Street Corporation	0.3934	0.0350	0.0654
SunTrust Banks, Inc	0.8538	0.0414	0.1069
The Bank of New York Mellon	0.4361	0.0218	0.0559
The Goldman Sachs Group, Inc	0.6661	0.0559	0.1063
The PNC Financial Services Group, Inc	0.8154	0.0690	0.1298
U.S. Bancorp	0.7893	0.0472	0.1073
Wells Fargo & Company	0.7364	0.0589	0.1143
Zions Bancorporation	0.7995	0.0707	0.1301

Table 1: Calibration

## 4.2 Calibration of the model and equilibrium computation

For all banks involved in the 2015 CCAR, we collected from Bloomberg the total equity, that is Tier 1 + Tier 2 equity, the risk-weighted assets and total assets as of fiscal year end 2014. We display our data in Table (3) in appendix. It is important to note that all these data can also be retrieved from the public annual report of each bank, available on their website. For the empirical analysis, we make the assumption that the implied aggregate risk weight for each bank is given by equation (7).

Within our model, the total equity Tier 1 plus Tier 2 corresponds to  $E_{i,t}$  and the risk-weighted assets to  $\alpha_i q_i P_t$ . Moreover, the minimum ratio for total equity over risk-weighted assets is  $\theta_{min} = 8\%$ . Following equations (7), (12) and (13), it is straightforward to calibrate the values of  $\alpha$ ,  $\Delta^{sale}$  and  $\Delta^{fail}$  for each bank. We display these calibration results in Table (1). For example, the large international bank JP Morgan Chase has total equity of \$206 billion, risk-weighted assets of \$1,619 billion and total assets of \$2,572 billion. We find an implied risk-weight  $\alpha = 63\%$ , a fire sale threshold  $\Delta^{sale} = 3.15\%$  and an insolvency threshold  $\Delta^{fail} = 8.03\%$ . This means that if JP Morgan Chase's assets lose more than 8.03%, the bank will be insolvent. If assets lose more than 3.15%, but less than 8.03%, JP Morgan Chase will remain solvent but with a capital ratio below 8%. It will thus have to engage in fire sales to bring back its capital ratio above 8%. From Table (1), one can see that  $\underline{\Delta}^{sale} = 2.18\%$  and corresponds to the sale threshold of The bank of New York Mellon. In the same vein,  $\overline{\Delta}^{fail} = 14.94\%$  corresponds to the insolvency threshold of Discover Financial services.

Once the relevant quantities of our model have been calibrated, i.e., the implied weight and the two thresholds of each bank, we consider the smallest Nash equilibrium under different scenarios (shock/market impact). In practice, while the market depth can be estimated using for instance daily market data, e.g., [Kyle and Obizhaeva, 2016], we here consider a set of possible market depth, ranging from very low to large ones. In what follows, the price impact is measured by  $\frac{Q_{tot}}{\Phi}$ . To explicitly compute the Nash equilibrium, we make use of a fixed-point algorithm<sup>10</sup> that we somehow adapt to our specific framework. It is important to point out that our approach is based on calibration and not on econometrics, see [Bajari et al., 2013] for an overview of game theory and econometrics. The parameters are not estimated statistically but are rather implied (or calibrated) from the data contained in the annual reports of banks using our model. It is thus similar in the spirit to the way an

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<sup>10</sup>See also [Echenique, 2007] for the specific case of games with strategic complementarities.

implied volatility is computed from the observed price using the Black and Scholes model or to the way an implied default probability is computed from the observed CDS spread using an intensity model. The main advantage of our approach is that it is transparent and simple to reproduce.

### 4.3 Empirical results

We now examine empirically the impact of an exogenous shock on this banking system made of 30 US bank holding companies. We shall first consider liquidation and convexity effects and we shall then discuss contagion.

**Liquidation and convexity effect.** Let us first discuss the case where asset market depth/liquidity is infinite, that is  $\Phi = \infty$  or, equivalently,  $\frac{Q_{tot}}{\Phi} = 0$ . In Figure 4.3, the blue line displays the dollar size of fire sales in the banking system (Y axis) as a function of the shock size  $\Delta$  (X axis). As expected, when  $\Delta \leq \underline{\Delta}^{sale} = 2.18\%$ , all banks remain solvent and with a regulatory capital ratio above 8% despite the shock and there is no need to liquidate assets. Symmetrically, when  $\Delta \geq \overline{\Delta}^{fail} = 14.94\%$ , all banks become insolvent due to the size of the shock and are fully liquidated. Between those two thresholds, as opposed to the case of a single bank in which fire sales increase linearly with  $\Delta$ , we observe a *non-linear relationship* between fire sales and shock size in the case of a system with multiple banks. The volumes liquidated turn out to be highly convex for shock slightly above  $\underline{\Delta}^{sale} = 2.18\%$ . For instance, the volume of fire sales when  $\Delta = 6\%$  is equal to \$7,103 billion, which is much more than twice the volume of fire sales when  $\Delta = \frac{6\%}{2} = 3\%$ , which is equal to \$1,957 billion. This is due to the fact that the larger the shock, not only the larger the volumes liquidated by a given bank, but also the greater the number of banks engaging in fire sales.

From a regulatory perspective, our model enables to estimate such convexity effects and, more generally, the endogenous reaction of the banking system to an exogenous shock and hence anticipate potential destabilizing loops in a macro-prudential context. In order to avoid too large deleveraging phenomenon, the regulator could temporarily decrease capital requirements for banks after a large shock, that is decrease  $\theta_{min}$ . Our approach provides a simple theoretical framework together with numerical results that enable a regulator to assess the capital requirement relief needed for a given constraint. Consider once again the no price impact case and let us assume that the regulator would like (arbitrarily) to limit the volume of fire sales when  $\Delta = 6\%$  to \$6 billion. We know that if  $\theta_{min} = 8\%$ , liquidations

$$\mathcal{I}(\Delta, \delta = 100bps, \frac{Q_{tot}}{\Phi})$$

$\Delta_1$ to $\Delta_2$    $\frac{Q_{tot}}{\Phi}$	1%	3%	5%	6.75%	8.5%
5% to 6%	1	1.5	1.5	3	3
6% to 7%	2	3	7	7	0
7% to 8%	3.8	2.5	1.5	0	0.5
8% to 9%	0.5	1.25	1	0.25	0
9% to 10%	2.5	4	0	0	0
10% to 11%	1.25	0.25	0.25	0	0
11% to 12%	1.2	0.2	0	0	0

Table 2: Amplification effect

will amount to \$7,103 billion. We find numerically that the regulator should lower its capital requirement to  $\theta_{min} = 6.75\%$  in order to constrain the fire sales volume to \$6 billion when  $\Delta = 6\%$ . A similar analysis can be obviously done when the price impact is positive.

Consider now the case in which the price impact is positive. Figure 4.3 displays the volume of fire sales in the banking system as a function of  $\Delta$  for different values of  $\frac{Q_{tot}}{\Phi}$ . As expected, we find that the greater the size of the banking system compared to asset market depth, the greater the volume of fire sales following a given exogenous shock  $\Delta$ . Quite interestingly, one can see from Figure 4.3 that the excess fire sales due to market frictions are *maximal for intermediate shocks*. For instance, for an exogenous shock  $\Delta = 6\%$ , the volume of fire sales when  $\frac{Q_{tot}}{\Phi} = 3\%$  is doubled compared to the case without price impact. When banks have large positions compared to asset liquidity, market frictions are so important that any bank that starts deleveraging its portfolio is going to have a very large impact on asset prices and lead to a contagion of defaults (see green line in Figure 4.3). This extreme behavior is actually caused within our model by the fact that liquidations all take place at the same date  $t + 1$  whatever their size and price impact. As already said, in practice, to avoid this kind of problem in which each liquidated the same asset at the same time, regulators may temporarily decrease the required RBC.

**Amplification effect.** In table 2, we compute from table 4 (see the appendix) the possible amplification due a marginal shock (see equation 41). It is for intermediate shock size and price impact that the amplification effect is the highest. Consider once again an

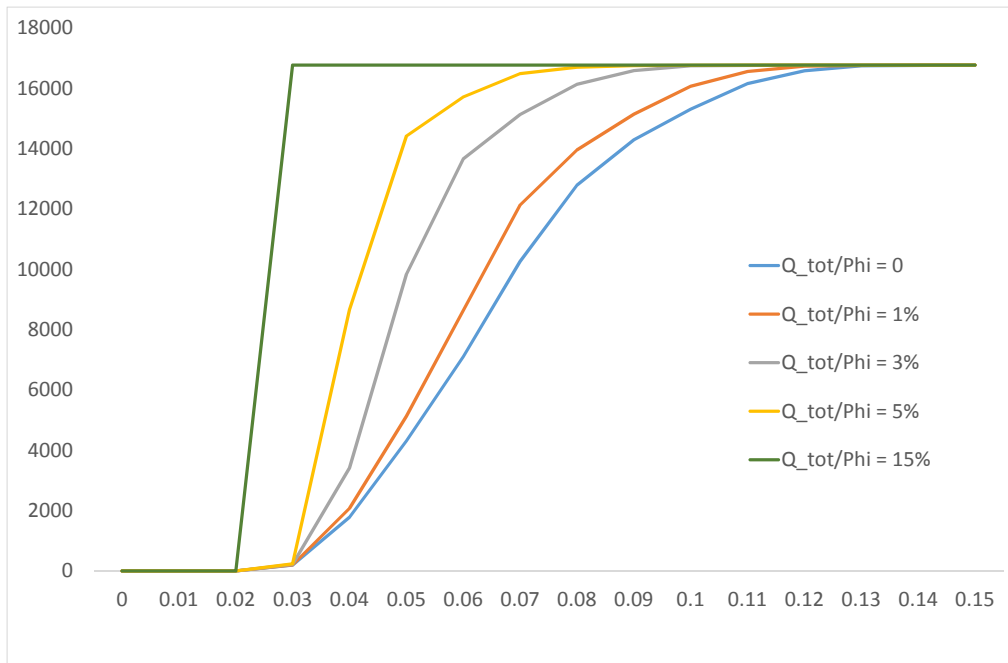


Figure 2: Volume of fire sales in Bn dollars in the banking system (Y axis) as a function of shock size  $\Delta$  (X axis)

initial shock  $\Delta = 6\%$  and assume that its size is increased by 100 basis points. Without price impact, the fraction of insolvent banks increases from 6.6% to 10% (see table 4). With a price impact of 5%, this fraction increases now from 40% to 63.33%, which means that the fraction of insolvent banks is multiplied by approximately by 7 i.e.,  $\mathcal{I} = \frac{0.633-0.4}{0.1-0.066} \approx 7$ . Table 2 provides the different values of the index  $\mathcal{I}$  in various scenarios and one can see that for moderate initial shock and moderate price impact, in general,  $\mathcal{I}$  is greater than one. However, when the price impact is high (say higher than 7%) and when the initial shock is large (say greater than 9%), there is no amplification effect because most of the banks are already insolvent for the initial shock. In such a case, when the size of the shock is increased by 100 bps, this marginal shock has virtually no consequence and this explains why  $\mathcal{I}$  is close to zero or even equal to zero.

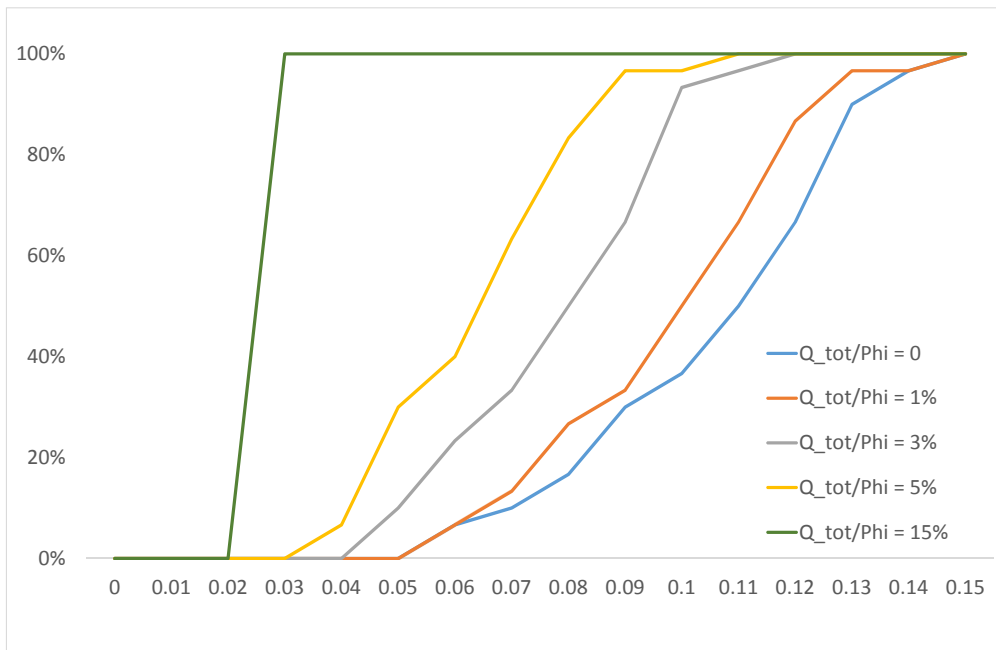


Figure 3: Proportion of insolvent banks (Y axis) as a function of shock size  $\Delta$  (X axis)

#### 4.4 What happens if GSIB capital surcharge as of 2016 was already fully implemented in 2014 ?

**On macroprudential regulation.** The purpose of macroprudential regulation is to limit the likelihood and costs of indirect contagion [Greenlaw et al., 2012], see also the policy paper of [Clerc et al., 2016]. In the academic literature, various macroprudential tools such as time varying capital, contingent capital or higher quality capital have indeed been considered [Hanson et al., 2011]. Few years ago, the Basel Committee proposed a methodology to quantitatively define a macroprudential instrument designed to assess the *capital surcharge* of a bank classified as global systemically important bank (GSIB). This methodology is based on a final score supposed to reflect the *systemic impact* of the bank, and is computed as a function of indicators such as the size and interconnections of the bank ([Board, 2017]). A bank classified as a GSIB will be required, depending upon its final score, to have a higher loss absorbency (HLA) expressed in percentage of the RWA. The final score is expressed in

basis points and the HLA is a piecewise constant function; as long as the final scores falls between two thresholds, typically  $x$  and  $x + 100$  bps, the HLA of the bank is constant and is equal to one of the five buckets 1%, 1.5%, 2%, 2.5%, 3.5% (in 2017, the top bucket 3.5% is still empty). This capital surcharge requirement began in January 2016 and will be fully implemented in 2019. The 2016 list of American banks classified as GSIBs can be found in a document published by the Financial Stability Board<sup>11</sup>.

**The situation as of 2014.** Consider a shock  $\Delta = 6\%$ . From table 1, only two banks, namely BMO Financial corp and The Bank of New York Mellon, have an insolvency threshold  $\Delta^{fail}$  lower than 6%. When there is no price impact, i.e.,  $\frac{Q_{tot}}{\Phi} = 0\%$ , only these two banks fail. Since there are 30 banks, the fraction of insolvent banks thus is equal to 6.66%, which is the number reported in table 4 (see the appendix) for this initial shock of 6%. When the price impact is now equal to 3%, i.e.,  $\frac{Q_{tot}}{\Phi} = 3\%$ , seven banks are actually insolvent so that the fraction of insolvent banks is equal to 23.3%. Among them, we find the well-known large international financial institutions Bank of America, HSBC North America Holding, JP Morgan Chase but we also find, beside BMO Financial corp and The Bank of New York Mellon, banks such as Santander Holding USA and State Street Corporation, whose total assets are much lower (respectively equal to 118 and 274 billion of dollars). It is interesting to note that out of the seven banks that are insolvent under our scenario (initial shock of 6%, price impact of 3%), six are indeed classified as GSIB. The subset of insolvent banks in our model thus is a good predictor of the subset of banks identified as GSIBs, including banks with a total value of the assets much lower than the well-known large international banks.

**The situation of 2014 assuming the 2016 GSIB capital surcharge.** It is now interesting to conduct our exercise: as a function of the price impact, what can be said of the fraction of insolvent banks at equilibrium if the capital surcharge would have been fully implemented in 2014 ? From the information contained in the 2016 list, the capital surcharge as a function of the RWA is equal to 2.5% for Citigroup and JP Morgan Chase, 2% for Bank of America and HSBC, 1.5% for Goldman Sachs and Wells Fargo, 1% for Bank of New York Mellon, Morgan Stanley, Santander Holding USA and State Street. To be very concrete, as of 2014, the RWA of Bank of America is equal to \$1,262 Billion (see table 3) and is allocated

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<sup>11</sup>See the website of the FSB, <http://www.fsb.org/wp-content/uploads/2016-list-of-global-systemically-important-banks-G-SIBs.pdf>



to the bucket 2%. As a result, the additional buffer is equal to \$25 Billion so that the new total capital is equal to \$187 Billion instead of \$162 Billion.

When the initial shock is equal to 6% and the price impact equal to 3%, beside BMO Financial corp and The Bank of New York Mellon, only State Street Corporation defaults, which means that only three banks are now insolvent, that is, 10% of the banks are insolvent. Quite interestingly, the well large international banks are able to survive at equilibrium. Compared to the no capital buffer case, the difference is important as the fraction of insolvent banks is reduced by more than 50%, i.e., from 23% to 10%.

## 5 Conclusion

In this paper, we developed a stylized framework of strategic macro stress test in which banks are hit by a common shock and may have to sell assets. This naturally leads to a game with strategic complementarities for which we showed the existence of a Nash equilibrium and the way to choose it when non-unique. We then explained how to calibrate the parameters of our model to public data and studied empirically the set of American banks that were part of the regulatory stress as of 2015 (CCAR), providing various comparative analyses as a function of shock size and/or asset liquidity. Finally we explained how our framework can be used to draw macro prudential regulatory measures such as the capital surcharge for GSIBs.

## 6 Appendix : proofs

### Proof of lemma 2

**Part 1.** We shall first prove the following Lemma:

**Lemma A 1** *Assume that  $\Delta < \Delta_i^{fail}$  and  $v_1 \leq v_2$ . Then, for all  $x_i \in [0, 1)$ ,  $\theta_{i,t+1}(x_i, v_1, \Delta) \geq \theta_{i,t+1}(x_i, v_2, \Delta)$ .*

**Proof.** Consider first the case in which, for a given  $x_i < 1$ ,  $\Delta_i^{fail} - \Delta(x_i, v_2) > 0$ . Since  $v_1 \leq v_2$  and since  $\Delta(x_i, v)$  is an increasing function of  $v$ , it thus follows that  $\Delta_i^{fail} - \Delta(x_i, v_1) > 0$ . It is not difficult to show that if  $v_1 \leq v_2$ , then,  $\theta_{i,t+1}(x_i, v_1, \Delta) \geq \theta_{i,t+1}(x_i, v_2, \Delta)$ . Assume now that for a given  $x_i < 1$ ,  $\Delta_i^{fail} - \Delta(x_i, v_2) \leq 0$  so that  $\theta_{i,t+1}(x_i, v_2) = 0$ . By definition, capital ratios are positive and we have  $\theta_{i,t+1}(x_i, v_1) \geq \theta_{i,t+1}(x_i, v_2)$  which concludes the proof of lemma 1  $\square$

We can now prove part 1 of lemma 2. When  $BR_i(v_2, \Delta) = 1$ , we clearly have  $BR_i(v_1, \Delta) \leq BR_i(v_2, \Delta)$ . Assume that  $BR_i(v_2, \Delta) < 1$  and note that this implies that  $\Delta < \Delta_i^{fail}$ .

- In the case where  $BR_i(v_2, \Delta) = 0$ , then we have  $\theta_{i,t+1}(0, v_2) \geq \theta_{min}$ . Given lemma 1, this means that  $\theta_{i,t+1}(0, v_1) \geq \theta_{min}$  and so  $BR_i(v_1, \Delta) = 0 \leq BR_i(v_2, \Delta)$ .
- In the case where  $0 < BR_i(v_2, \Delta) < 1$ , this implies that  $\theta_{i,t+1}(BR_i(v_2, \Delta), v_2) = \theta_{min}$ . Given lemma 1, this means that  $\theta_{i,t+1}(BR_i(v_2, \Delta), v_1) \geq \theta_{min}$  which implies that  $BR_i(v_1, \Delta) \leq BR_i(v_2, \Delta)$  and concludes the proof  $\square$

### Part 2.

The proof of part 2 is similar. One can show that for all  $v > 0$ , if  $\Delta_1 \leq \Delta_2$ , then, for all  $x_i \in [0, 1)$ ,  $\theta_{i,t+1}(x_i, v, \Delta_1) \geq \theta_{i,t+1}(x_i, v, \Delta_2)$  which implies part 2 of lemma 2.  $\square$

**Proof of Proposition 1** For an arbitrary shock  $\Delta$  and an arbitrary market depth  $\Phi > 0$ , from lemma 2, the best response  $BR_i(x_{-i})$  of each bank  $i = 1, \dots, p$  is an increasing function of  $x_{-i}$ . By letting  $f := \prod_{i=1}^p BR_i$ ,  $f$  is a non-decreasing function from  $X$  to  $X$ , where  $X = [0, 1]^p$  so that from Tarski's theorem,  $\mathcal{F}_\Delta$  is not empty and is moreover a complete lattice and this concludes the proof of existence. Note that although the best response is unique in our model, the existence result would work for a best response correspondence.

The proof that the subsets  $S^*$  and  $D^*$ , when non empty, are composed with consecutive integers relies on the two following lemma.

**Lemma A 2** *For all initial shocks  $\Delta > 0$  and all equilibria  $\mathbf{x}^* \in \mathcal{F}_\Delta$*

1. *if  $x_i^* < 1$ , then,  $\Delta(\mathbf{x}^*) < \Delta_i^{fail}$ .*
2. *if  $x_i^* = 1$ , then,  $\Delta(\mathbf{x}^*) \geq \Delta_i^{fail}$ .*

**Proof** Part 1. Consider a given equilibrium  $\mathbf{x}^* \in \mathcal{F}_\Delta$ . If  $x_i^* < 1$ , bank  $i$  is solvent, i.e.,  $\theta_{i,t+1}(x_i^*, x_{-i}^*) \geq \theta_{min}$ , then, its total capital must be positive;  $E_{i,t+1}(\mathbf{x}^*) > 0$ . From the equivalence provided in equation (25), it thus must be the case that  $\Delta(\mathbf{x}^*) < \Delta_i^{fail}$ . Part 2. Assume that the contrary is true, i.e., when  $\mathbf{x}^* := (x_i^* = 1, x_{-i}^*)$ ,  $\Delta(\mathbf{x}^*) < \Delta_i^{fail}$ . From equation (25), this means that the total capital  $E_{i,t+1}(\mathbf{x}^*) > 0$ . But then, since the total capital is positive for each  $x_i \in [0, 1]$ , given  $x_{-i}^* \in [0, 1]^{p-1}$ , there exists  $x_i^* < 1$  such that  $\theta_{i,t+1}(x_i^*, x_{-i}^*) = \theta_{min}$ , and this contradicts the optimality of the best response  $x_i^* := BR_i(x_{-i}^*) = 1$ .  $\square$

**Lemma A 3** *For all initial shocks  $\Delta > 0$  and all equilibria  $\mathbf{x}^* \in \mathcal{F}_\Delta$*

1. *if there exists  $0 \leq i_1 \leq p$  such that  $x_{i_1}^* < 1$ , then  $x_i^* < 1$  for all  $i \geq i_1$*
2. *if there exists  $0 \leq i_0 \leq p$  such that  $x_{i_0}^* = 1$ , then  $x_i^* = 1$  for all  $i \leq i_0$*

**Proof** Part 1 Consider a given equilibrium  $\mathbf{x}^* \in \mathcal{F}_\Delta$ . Assume that  $x_{i_1}^* < 1$  and consider  $i \geq i_1$ . Since  $x_{i_1}^* < 1$ , this means that the equity of bank  $i_1$  at equilibrium is positive, that is  $\Delta_{i_1}^{fail} - \Delta(\mathbf{x}^*) > 0$ . Since  $\Delta_i^{fail} \geq \Delta_{i_1}^{fail}$ , it thus follows that for each  $i \geq i_1$ ,  $\Delta_i^{fail} - \Delta(\mathbf{x}^*) \geq 0$ . Part 2. Assume that  $x_{i_0}^* = 1$  and consider  $i \leq i_0$ . As  $x_{i_0}^* = 1$ , this means that the equity of bank  $i_0$  at equilibrium is equal to zero, that is  $\Delta_{i_0}^{fail} - \Delta(\mathbf{x}^*) \leq 0$ . Since  $\Delta_i^{fail} \leq \Delta_{i_0}^{fail}$ , it thus follows that for each  $i \leq i_0$ ,  $\Delta_i^{fail} - \Delta(\mathbf{x}^*) \leq 0$   $\square$

That  $D^*$  and  $S^*$ , when non empty, are composed with consecutive integers is a direct consequences of the above lemmas and note that the existence of  $i^* := i(\mathbf{x}^*)$  depends on the equilibrium  $\mathbf{x}^*$ , i.e.,  $i(\mathbf{x}^*) \leq i(\mathbf{y}^*)$  if  $\mathbf{x}^* \leq \mathbf{y}^*$   $\square$

**Proof of Proposition 2** Given the assumption, we know that for all  $1 \leq i \leq p$ ,  $0 < x_i^* < 1$ , which means that the capital ratio of each bank at date  $t + 1$  is equal to  $\theta_{min}$ . From equation (24),  $\theta_{i,t+1}(\cdot) = \theta_{min}$  for all  $i$  is equivalent to

$$\frac{\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1 - \Delta)}{\alpha_i(1 - x_i^*)(1 - \Delta) \left(1 - \frac{Q^*}{\Phi}\right)} = \theta_{min} \quad i = 1, 2, \dots, p. \quad (42)$$

and this implies that:

$$q_i - x_i^* q_i = \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1 - \Delta)\right] q_i}{\theta_{min} \alpha_i (1 - \Delta) \left(1 - \frac{Q^*}{\Phi}\right)} \quad i = 1, 2, \dots, p. \quad (43)$$

and, summing for  $i = 1$  to  $p$ , we obtain that:

$$\sum_{i=1}^p q_i - Q^* = \sum_{i=1}^p \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1 - \Delta)\right] q_i}{\theta_{min} \alpha_i (1 - \Delta) \left(1 - \frac{Q^*}{\Phi}\right)} \quad (44)$$

Equation (44) remains an implicit function as  $Q^*$  appears both on the lfs and rhs. Writing (44) as  $Q^* + \sum_{i=1}^p \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1 - \Delta)\right] q_i}{\theta_{min} \alpha_i (1 - \Delta) \left(1 - \frac{Q^*}{\Phi}\right)} - \sum_{i=1}^p q_i = 0$ , it reduces to a quadratic equation in  $Q^*$ . It thus follows that  $Q^*$  is the root of a quadratic equation that actually depends on  $\Phi$  in a smooth manner, which implies that  $Q^*$  is a smooth function (i.e., regular) of  $\Phi$ . Recall that when  $\Phi = \infty$ , we know that  $Q^* = \underline{Q}^* = \sum_{i=1}^p x_i^* q_i$  so that, from equation (33), we have:

$$\underline{Q}^* = \sum_{i=1}^p q_i - \sum_{i=1}^p \frac{(\Delta_i^{fail} - \Delta) q_i}{\theta_{min} \alpha_i (1 - \Delta)} \quad (45)$$

Using the fact that  $Q^* \geq \underline{Q}^*$  and that  $Q^*$  is a smooth function of  $\Phi$  such that  $Q^*$  tends to  $\underline{Q}^*$  when  $\Phi$  tends to infinity, we thus can write:

$$Q^* = \underline{Q}^* \left(1 + \frac{\gamma}{\Phi}\right) + o\left(\frac{1}{\Phi}\right) \quad (46)$$

for some positive  $\gamma$ . Reintroducing the expression of  $Q^*$  given in equation (46) into (44), and neglecting terms of  $o\left(\frac{1}{\Phi^2}\right)$ , it is not difficult (but cumbersome) to identify the term  $\gamma$  to find equation (40). Using equation (43), we then find equation (39)  $\square$

## Data from Bloomberg as of 2014.

Bank	Total Capital	RWA	Total Assets	RBC
Ally Financial Inc	17294	130590	151631	0,132
American Express Company	20800	133300	159103	0,156
Bank of America Corporation	161623	1262000	2104534	0,147
BB&T Corporation	21381	143675	186834	0,149
BBVA Compass Bancshares, Inc	8254.2	64417.8	83152.4	0,128
BMO Financial Corp	31927	222931	588659	0,143
Capital One Financial Corporation	35886	237587	308167	0,151
Citigroup Inc	165454	1292605	1842181	0,128
Citizens Financial Group Inc	16781	105964	132857	0,158
Comerica Incorporated	7169	68273	69190	0,105
Discover Financial Services	12418	72744.9	83126	0,171
Fifth Third Bancorp	16895	117878	138706	0,143
HSBC North America Holdings Inc	190730	1219800	2634139	0,156
Huntington Bancshares Incorporated	7388	54479	66298	0,136
JPMorgan Chase &Co	206594	1619287	2572274	0,128
KeyCorp	11824	85100	93821	0,139
M&T Bank Corporation	11767.3	77365.6	96685.5	0,152
Morgan Stanley	74972	456008	801510	0,164
MUFG Americas Holdings Corporation	14246	96663	113623	0,147
Northern Trust Corporation	9449.2	62896.9	109946.5	0,150
Regions Financial Corporation	15070	98974	119563	0,152
Santander Holdings USA, Inc	9872.6	66751.9	118457.4	0,148
State Street Corporation	17914	107827	274119	0,166
SunTrust Banks, Inc	20338	162500	190328	0,125
The Bank of New York Mellon	21556	168028	385303	0,128
The Goldman Sachs Group, Inc	90978	570313	856240	0,160
The PNC Financial Services Group, Inc	44782	281379	345072	0,159
U.S. Bancorp	43208	317705.9	402529	0,136
Wells Fargo & Company	192900	1242500	1687155	0,155
Zions Bancorporation	7443.301	45738	57208.8	0,163

Table 3: All quantities except the RBC are in million of dollars

**Fraction of insolvent banks at equilibrium as a function of  $\Delta$  and  $\Phi$**

$\Delta \parallel \frac{Q_{tot}}{\Phi}$	0	1%	3%	5%	6.75%	8.5%	10%	11.75%	15%
0.01	0	0	0	0	0	0	0	0	0
0.02	0	0	0	0	0	0	0	0	0
0.03	0	0	0	0	0,33	0	0	0	1
0.04	0	0	0	0.066	0,53	0.66	0.96	1	1
0.05	0	0	0.1	0.3	0.73	0.86	0.96	1	1
0.06	0.066	0.066	0.233	0.4	0.96	0.96	1	1	1
0.07	0.1	0.133	0.333	0.633	0.96	0.96	1	1	1
0.08	0.166	0.266	0.5	0.833	1	1	1	1	1
0.09	0.3	0.33	0.666	0.966	1	1	1	1	1
0.1	0.366	0.5	0.933	0.966	1	1	1	1	1
0.11	0.5	0.666	0.966	1	1	1	1	1	1
0.12	0.666	0.86	1	1	1	1	1	1	1
0.13	0.9	0.966	1	1	1	1	1	1	1
0.14	0.966	0.9667	1	1	1	1	1	1	1
0.15	1	1	1	1	1	1	1	1	1

Table 4: Fraction of insolvent banks

**Best response of each bank when  $\Delta = 6\%$ .**

Bank	$\frac{Q_{tot}}{\Phi} = 0$	$\frac{Q_{tot}}{\Phi} = 1\%$	$\frac{Q_{tot}}{\Phi} = 3\%$	$\frac{Q_{tot}}{\Phi} = 5\%$	$\frac{Q_{tot}}{\Phi} = 15\%$
Ally Financial Inc	0.16	0.23	0.54	0.84	1
American Express Company	0	0	0.26	0.55	1
Bank of America Corporation	0.62	0.73	1	1	1
BB&T Corporation	0.05	0.13	0.48	0.81	1
BBVA Compass Bancshares, Inc	0.32	0.40	0.76	1	1
BMO Financial Corp	1	1	1	1	1
Capital One Financial Corporation	0.02	0.10	0.45	0.78	1
Citigroup Inc	0.43	0.52	0.91	1	1
Citizens Financial Group Inc	0	0	0.30	0.61	1
Comerica Incorporated	0.41	0.47	0.75	1	1
Discover Financial Services	0	0	0	0.28	1
Fifth Third Bancorp	0.03	0.10	0.41	0.71	1
HSBC North America Holdings Inc	0.64	0.78	1	1	1
Huntington Bancshares Incorporated	0.16	0.24	0.57	0.87	1
JPMorgan Chase &Co	0.57	0.67	1	1	1
KeyCorp	0.03	0.09	0.39	0.66	1
M&T Bank Corporation	0	0.05	0.38	0.69	1
Morgan Stanley	0.21	0.32	0.81	1	1
MUFG Americas Holdings Corporation	0	0.04	0.36	0.65	1
Northern Trust Corporation	0.39	0.50	0.99	1	1
Regions Financial Corporation	0	0.01	0.33	0.63	1
Santander Holdings USA, Inc	0.44	0.56	1	1	1
State Street Corporation	0.81	0.98	1	1	1
SunTrust Banks, Inc	0.27	0.34	0.66	0.95	1
The Bank of New York Mellon	1	1	1	1	1
The Goldman Sachs Group, Inc	0.07	0.16	0.57	0.95	1
The PNC Financial Services Group, Inc	0	0	0.26	0.56	1
U.S. Bancorp	0.20	0.28	0.62	0.94	1
Wells Fargo & Company	0.01	0.10	0.46	0.81	1
Zions Bancorporation	0	0	0.24	0.54	1

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