Trump trumps Bush

Rafael Treibich[1] and Martin Van der Linden[2]

[1]University of Southern Denmark
[2]Utah State University

December 5, 2017

Abstract

In the United States, the legitimacy of presidents who do not obtain a majority of the popular vote is often questioned. Debates on electoral legitimacy tend to revolve around the relative importance of the Electoral College and the popular vote. We develop a theory of electoral legitimacy judgments when legitimacy depends on these two factors. Under natural restrictions on these judgments, the legitimacy of some candidates can be unambiguously compared. In particular, we show that D. Trump's election was more legitimate than G.W. Bush's 2000 election. We also show that Trump's election remains one of the most contentious in history.

1 Introduction

In the United States (U.S.), the possibility for the elected president to have lost the popular vote is a well-known feature of the Electoral College rule (EC).[1] Two recent examples are the elections of D. Trump in 2016 (v. H. Clinton) and G.W. Bush in 2000 (v. A. Gore). The two elections were followed by contentious debates on the legitimacy of the newly elected president. These debates demonstrate that a candidate victory on election day does not necessarily result in citizens viewing the candidate as electorally legitimate. The room given to these debates in the media and in the public discourse also illustrates the political importance of such subjective legitimacy judgments in the population.

In the case of U.S. presidential elections, debates about the legitimacy of an elected candidate typically revolve around the relative importance of the Electoral College and the popular vote. Supporters of the new president's legitimacy often argue that the EC plays an important role in maintaining the federal character of the U.S. (Sabato, 2007).

[1] When combined with winner-take-all systems at the state-level. By "loosing the popular vote", we mean receiving less than a plurality of the nationwide popular vote.

Opponents on the other hand incriminate the EC for violating the one-person one-vote principle (Edwards, 2004).[2]

Questions of legitimacy may also arise when, despite winning the popular vote, the winning candidate does not receive the *absolute* majority of the nationwide popular vote. In 1992, for example, Senate Minority Leader Bob Dole questioned the legitimacy of newly elected president B. Clinton, on the ground that, despite winning the election, B. Clinton only secured 43% of the popular vote (B. Clinton nevertheless won the popular vote, as his runner-up H.W. Bush only obtained 37.4% of the popular vote).[3] While a winning candidate losing the popular vote has occurred only 4 times in U.S. presidential history, a winning candidate not securing the absolute majority of the popular vote has occurred 14 times in the 36 post-1876 elections.[4]

Despite the importance of subjective legitimacy judgments in the public debate, no model of these judgments have been proposed that would yield clearcut comparisons of the legitimacy of different candidates following an election. In this paper, we develop a simple but elegant model of subjective legitimacy judgments which enables such comparisons in the context of U.S. presidential elections.

In our model, elections features two top candidates (a democrat and a republican) who receive more of both the popular and the electoral vote than all other candidates (independents). There is a set of political observers whose opinion matters to determining the electoral legitimacy of all candidates. Observers form binary judgments about the legitimacy of candidates following an election, viewing candidates as either legitimate or illegitimate. These judgments are based on the *electoral record* for that election, which specifies the share of the popular and electoral votes received by all candidates. In this context, observers' judgments can be interpreted as the observers' views on the voting rule they think *should* have been used to determine the winning candidate.[5] We then aggregate these judgments in a particularly robust manner to compare the overall legitimacy of candidates within and across elections.

We impose three kinds of restriction on legitimacy judgments. First, we require legitimacy judgments to match likely features of empirical legitimacy judgments that people may express in surveys. For example, we impose that legitimacy be *monotonic*, meaning that the legitimacy of a candidate cannot decrease when votes are transferred from other

[2] Opponents and supporters of the EC rely on a variety of other arguments, such as the emphasis the EC puts on swing states (Edwards, 2004), or the purported tendency of the EC to protect the electoral impact of rural areas compared to that of large cities (Sandlin, 2000; Paul, 2000).

[3] By "winning the popular vote", we mean receiving more than a plurality of the nationwide popular vote.

[4] The Electoral College system underwent a number of changes since its inception. The last major changes came with the adoption in 1868 of the Fourteenth Amendment to the U.S. Constitution, whose Section 2 guarantees that Electors be nominated by states based on a vote among eligible state inhabitants (as opposed to a vote of the state legislature). We focus on post-1876 elections, the election of 1876 being the first in which the elected candidate (R. Hayes) did not win the popular vote.

[5] A candidate being legitimate if she or he would have been elected under the voting rule that the observer favors, and *i*llegitimate if the candidate would not have been elected under the voting rule that the observer favors.

candidates to that candidate.

Second, we impose restrictions that may not be satisfied by empirical legitimacy judgments, but are required to isolate *electoral* legitimacy from other sources of political legitimacy. For example, we require legitimacy judgements to be *anonymous* meaning that legitimacy should only depend on the electoral record itself, and not on the identity of candidates.

Finally, we consider two restrictions capturing opposing approaches to electoral legitimacy. Under the first "absolutist" approach, a candidate's legitimacy depends only on her or his *own* electoral record, and not on the record of the other candidates running in the same election. This approach is embodied in an *independence* axiom which forces legitimacy judgements to depend only on a candidate's *own* shares of popular and electoral vote. In contrast, under the second "relativist" approach, a candidate's legitimacy also depends on the electoral record of other candidates running in the same election. This approach is embodied in a *relativity* axiom that forces legitimacy judgements for the two top candidates to depend on these candidates' *relative* shares of popular and electoral vote. For example, while an absolutist might question Bill Clinton's legitimacy for only winning 43% of the popular vote in 1992, a relativist might instead argue that Bill Clinton would have fared better if it were not for the third independent candidate Ross Perot (who garnered almost 19% of the popular vote) and should therefore be considered legitimate.

We show how combining a set of weak restrictions (including Monotonicity and Anonymity) with Independence forces legitimacy judgments to (i) be linear with respect to the shares of popular and electoral vote and (ii) consider legitimate (illegitimate) any candidate who gets a majority (minority) of both the popular and electoral votes. Similarly, combining the same set of restrictions with Relativity forces legitimacy judgments to (i') be linear with respect to the *relative* shares of popular and electoral vote and (ii') consider legitimate (illegitimate) any candidate who wins (loses) both the *relative* popular and electoral votes.

When legitimacy judgments can be represented by such linear functions, we show that the legitimacy of some U.S. presidential candidates is supported by a larger fraction of observers than the legitimacy of other candidates, *independent* of the particular distribution of legitimacy judgments among observers.[6] In this sense, our comparisons are robust because they do not rely on information regarding the observers' actual legitimacy judgements.

Some of our comparisons are doubly robust as they depend neither on the particular distribution of weights that observers put on the popular vote relative to the electoral vote (i.e., the slopes of observers' linear legitimacy judgments), *nor* on the assumption that

[6] Many of these comparisons are *not* a mere consequence of Monotonicity and follow from a more subtle interaction between the natural restrictions we impose on legitimacy judgments.

observers have absolutist or relativistic judgments. For example, we find that Trump's election was more legitimate than G.W. Bush 2000's election under both the absolutist and the relativist approach. However, Donald Trump's election remains among the most contentious from the point of view of electoral legitimacy: All but three of U.S. presidents since 1876 (namely R. Hayes, G.W. Bush in 2000, and R. Nixon in 1968) are found to be electorally *more* legitimate than Donald Trump.[7]

Other comparisons are sensitive to fixing the judgment type in the observers' population. For example, under the absolutist approach, all elected presidents post-1876 were more legitimate than their runner-up. In particular, under the absolutist approach, Donald Trump's electoral record makes him more electorally legitimate than Hillary Clinton. In the relativist approach, the same is only true for elected candidates who also won the popular vote. In the presence of relativist observers, the legitimacy of the four candidates who won without winning the popular vote (Hayes, Harrison, G.W. Bush, and Trump) cannot be compared with the legitimacy of their runner-up without further assumptions on the distribution of legitimacy judgments. In particular, under the relativist approach, Trump's legitimacy cannot be uncontroversially compared with the legitimacy of Hillary Clinton. However, we show that it would take a particularly skewed distribution of legitimacy judgments for Hillary Clinton to be considered more legitimate than Donald Trump.

Related literature. *Political* legitimacy is among the most widely discussed issues in political science and political philosophy. From a normative perspective, prominent authors like Rawls, Buchanan, Bentham or Habermas have debated the acceptable sources of political legitimacy (see Peter, 2017, for a review). From a more descriptive perspective, authors since Weber have attempted to explain subjective judgments on the legitimacy of political authority. Measures of political legitimacy based on desirable characteristics of political systems or public opinion have been previously developed (see Weatherford, 1992, for a classical discussion).

The focus of this paper is on *electoral* rather than *political* legitimacy. In this respect, our work relates to the sizable empirical literature studying the determinants of support for the elected candidates following an election.[8] In line with this literature, we ground our comparisons of electoral legitimacy in *subjective* judgments in the population. Instead of directly eliciting legitimacy judgments through surveys, we rely on a structural model of subjective legitimacy judgments. One motivation for doing so is our desire to isolate *electoral* legitimacy from other sources of political legitimacy. Even carefully crafted

[7] Again, this last result is true regardless of the distribution of the slopes of linear legitimacy judgments, and under both the absolutist and the relativist approach.

[8] The perception of an election's *fairness* is also often studied. Example of determinants studied include the consensual or majoritarian character of the political system (Anderson and Guillory, 1997), partisan cleavages (Craig et al., 2006), the propensity of elections to generate turnovers (Moehler and Lindberg, 2009), the competitiveness of the campaign (Wolak, 2014), and the quality of election administration (as measured, e.g., by wait time for voting Bowler et al., 2015).

survey questions are bound to incorporate noise from non-electoral sources of legitimacy.

To eliminate this noise, we force legitimacy judgments to be based on a candidate's electoral record, rather than her or his identity. Specifically, we are interested in judgments that are based on the two variables that crystallize debates on electoral legitimacy following contentious presidential elections in the U.S.: support from the states (as measured by the number of grand electors), and popular support (as measured by the popular vote). One of our contributions is to show that, once interferences from non-electoral sources of legitimacy are neutralized (and other natural restrictions are imposed), legitimacy comparisons can be reached that are independent of the actual distribution of subjective judgments in the population. In other words, we reach meaningful legitimacy comparisons that hold for *any* estimation of our structural model.

Our formal analysis — in particular its reliance on axiomatic arguments — is largely inspired by voting theory. The EC, and federal elections more generally, have been widely studied in voting theory. Special attention has been devoted to the propensity of the EC and other two-tier electoral rules to induce elections in which the winner looses the popular vote (May, 1948; Merrill, 1978; Nurmi, 1999; Laffond and Laine, 2000; Feix et al., 2004; Chambers, 2008; Miller, 2012), and on the differences in voter's electoral "power" implied by these rules (Banzhaf ill, 1968; Merrill, 1978; Sterling, 1981; Warf, 2009).

Unlike voting theory which aims at identifying desirable voting rules, we compare the legitimacy of election candidates given the voting rule *in place*. Many voting rules can select winners that are not supported by more than half of the voters. As the EC illustrates, some voting rules can even select winners that received less than a plurality of the votes. The question of the winner's electoral legitimacy then naturally emerges. In our model, legitimacy judgments in the population can be interpreted as preferences on the voting rule that *should have* been employed to determine the election winner. Therefore, our paper can also be viewed as tackling the issue of preferences *over voting rules*, and how these preferences interact with the chosen voting rule to legitimize or delegitimize the winner that the voting rule selects.

2 Model

We are interested in the situation that prevails *after* presidential elections in a federal system with two or more candidates. Although our model can be applied to other elections with well-defined measures of popular and state-based electoral support, our focus is on U.S. presidential elections. As a result, we talk of the *electoral* vote to refer to support from the states, and an election is characterized by a set of candidates each receiving a proportion of the electoral and popular votes.

Formally, candidates to an election are elements of a (finite) set of candidates $\mathbb{K}$.

An **election** is a triple $e := (K, p, g)$, where $K \subset \mathbb{K}$ is the set of candidates, $p := (p_c)_{c \in K} \in \Delta(K)$ is the list of shares of the popular vote and $g := (g_c)_{c \in K} \in \Delta(K)$ is the list of shares of the electoral vote. A pair (p, g) is called an **electoral record** and (p_c, g_c) is **candidate c's electoral record**. Although our application is the EC where the proportion of Electors supporting c is determined through a winner-take-all system based on the partition of voters into states,[9] we leave open the question of the relationship between p and g.

Throughout the analysis we assume a bipartisan environment whereby two candidates (a democrat and a republican) always get a larger share of both the popular and the electoral vote than all other candidates (independents). Formally, for any election $e = (K, p, g)$, there are two top candidates $d, r \in K$ for whom $p_r, p_d > p_c$ and $g_r, g_d > p_c$ for all $c \in K \backslash \{r, d\}$.[10]

For any election $e = (K, p, g)$, we define the reduced election $\hat{e}$ as the hypothetical election obtained from e by discarding all independent candidates (and their votes)

$$\hat{e} = \left(\{d, r\}, \frac{p_{d,r}}{p_d + p_r}, \frac{g_{d,r}}{g_d + g_r} \right).$$

We denote by $\hat{p}$ and $\hat{g}$ the *relative* shares of popular and electoral votes

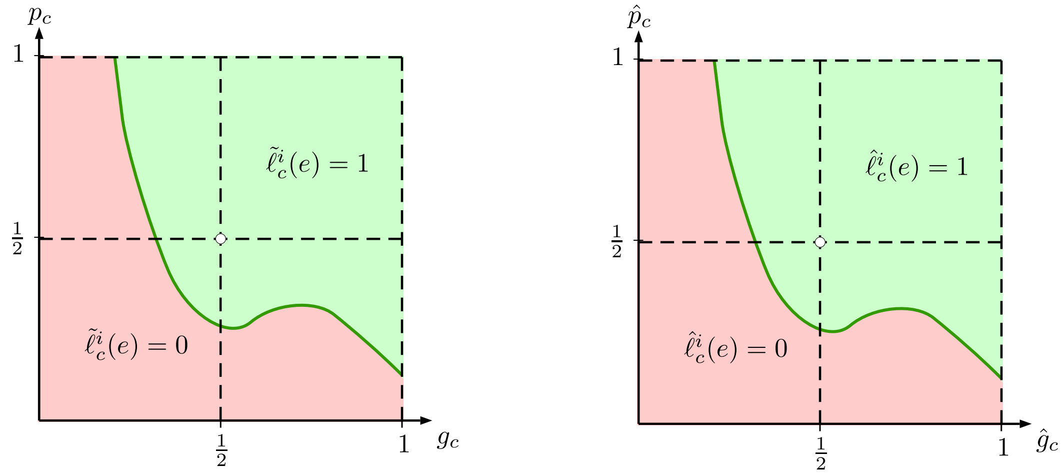
$$\hat{p} = \frac{p}{p_d + p_r} \quad \text{and} \quad \hat{g} = \frac{g}{g_d + g_r}.$$

An election is **absolute (relative) contentious** if the candidate who won the election did not secure the majority (plurality) of the popular vote. Following the popular terminology, we say that a candidate c **won the popular vote** if he obtained the plurality of the popular vote, i.e., $p_c > p_{c'}$ for all $c' \in K \backslash \{c\}$.

We are not interested in the winner of the election *per se*. Instead, we want to compare the political legitimacy of the *two* major candidates within and across elections. As suggested by the debates that followed the 2000 and 2016 U.S. presidential elections, a victory on election day does not necessarily result in a unanimous agreement on the legitimacy of the elected candidate. Some supporters of the losing candidate may accept the elected candidate as legitimate, whereas others will reject her or him. Some supporters of the winning candidate may also believe that, in spite of winning the election, the elected candidate did not secure a voting record that makes her or him electorally legitimate.

Let N be the population of (political) observers whose opinion is relevant to determining the political legitimacy of candidates. For any election $e = (K, p, g)$, each observer $i \in N$ has a binary judgment about the legitimacy of each of the candidates in K after election e. The judgment of observer i about candidate $c \in K$ is determined by the **legitimacy function** $\ell^i(.)$. A legitimacy function $\ell^i(\cdot)$ associates to every election e a vector

[9] With the exception of Maine and Nebraska who allocate their grand electors in a more proportional manner.
[10] The two top candidates d and r need not be the same in different elections.



(a) An absolute legitimacy function $\tilde{\ell}$.

(b) A relative legitimacy function $\hat{\ell}$.

Figure 1

$\ell^i(e) = (\ell^i_c(e))_{c \in K}$ in $\{0,1\}^{\#K}$, where $\ell^i_c(e)$ equals 1 (0) if i deems candidate c legitimate (illegitimate) for election $e = (K, p, g)$.

Figure 1 illustrates examples of legitimacy functions. The legitimacy function in Figure 1(a) is *absolute* in the sense that it depends only on the absolute shares of popular vote p_c and electoral vote g_c. The legitimacy function in Figure 1(b) is *relative* in the sense that it depends on the relative shares of popular vote $\hat{p}_c$ and electoral vote $\hat{g}_c$.

Note N is fixed, i.e., the same population of observers assesses all elections of interest. In particular, N needs not coincide with the population of voters who voted in the elections that are being compared. Population N can, for example, be understood as a panel of observers who *retrospectively* assess the legitimacy of candidates in past elections.

3 Legitimacy functions

3.1 Axioms

As debates following contentious U.S. elections suggest, observers typically differ in their legitimacy judgments about candidates. In particular, these judgments need not agree with the voting rule in place. In the U.S., there seems to be a divide between observers who favor the electoral vote and those who favor the popular vote as a source of legitimacy. Whether they favor the electoral of the popular vote, observers probably admit that certain trade-offs between the popular and the electoral vote are acceptable. For example, most observers would likely think that a candidate who secures 99.9% of the electoral vote and 49.9% of the popular vote is legitimate. In what follows, we look for reasonable restrictions on legitimacy judgments that more precisely determine the form these trade-offs can take.

First, it is natural to restrict attention to legitimacy functions that are *monotonic* with respect to the popular and electoral vote.

Definition 1 (Monotonicity)**.** *For any* $e = (K, p, g)$, $e' = (K, p', g')$ *and any* $c \in K$*, if* $p'_c \geq p_c$ *and* $g'_c \geq g_c$*, then* $\ell^i_c(e) = 1$ *implies* $\ell^i_c(e') = 1$*.*

Second, we require that the legitimacy function be *convex*. Convexity encompasses a notion of compromise. Although observers may disagree on the relative importance of the popular and electoral vote, they agree that a candidate tends to be more legitimate if she is supported by a mix of popular electors and grand electors (as opposed to being supported by a large share of one, but a small share of the other). Convexity is a reasonable restriction when it comes to judgments about winners.[11] It is much less reasonable to require that judgments about *losers* be convex.[12] Therefore, we only require the former.

Definition 2 (Convexity)**.** *For any* $e = (K, p, g)$, $e' = (K, p', g')$*, any* $c \in K$ *and any* $\lambda \in [0, 1]$*, if* $\ell^i_c(e) = \ell^i_c(e') = 1$*, then* $\ell^i_c(\lambda e + (1 - \lambda)e') = 1$*.*

Third, the legitimacy functions are *anonymous* in the sense that they depend on the candidates' electoral records and not on their identities. That is, legitimacy functions represent *impartial* evaluations of the candidates' electoral legitimacy. Two candidates must be equally legitimate if they face the same electoral record, even if the observer dislikes one of the two candidates, or if she thinks that a candidate is illegitimate for reasons that are independent from the electoral record.

Definition 3 (Anonymity)**.** *For any* $e = (K, p, g)$, $e' = (K', p', g')$*, if there exists a bijection* $\pi : K \mapsto K'$ *such that for any* $c \in K$*,* $p_c = p'_{\pi(c)}$ *and* $g_c = g'_{\pi(c)}$*, then for any* $c \in K$*,* $\ell^i_c = \ell^i_{\pi(c)}(e')$*.*

Fourth, the legitimacy functions are *resolute* in the sense that, for any election *with two candidates*, one and only one of the candidates is considered legitimate. Resoluteness embodies the idea that legitimacy judgments correspond to the observers' opinions of who *should* have won the election given the electoral record. For elections with more than two candidates, observers could believe that, given the electoral record, none of the candidates should have won the election (e.g., if each candidate secures exactly a third of both the popular and the electoral vote). In this case, Resoluteness is silent and allows observers to consider that none of the candidate is legitimate. For two-candidate elections, however,

[11] For example, suppose an observer considers that a candidate with 49.9% of the electoral vote and 99.9% of the popular vote is legitimate, and that a candidate with 99.9% of the electoral vote and 49.9% of the electoral vote is also legitimate. Then it seems reasonable to require that the observer also consider a candidate with 74.9% of both votes as legitimate (74.9 = (49.9 + 99.9)/2).

[12] For example, suppose an observer considers that a candidate with 45% of the electoral vote and 60% of the popular vote is illegitimate, and that a candidate with 60% of the electoral vote and 45% of the popular vote is also illegitimate. Then convexity of judgments about losers would require the observer to also conclude that a candidate with 52.5% of both votes is illegitimate (52.5 = (45 + 60)/2), which does not seem natural. A priori, there is no reason to prevent an observer from thinking that more than 45% of both votes is required for a candidate to be legitimate, while at the same time considering that a candidate who secure 52.5% of both votes is legitimate.

indecision is harder to justify, and Resoluteness therefore requires observers to make up their mind about which of the two candidates they consider legitimate.

Definition 4 (Resoluteness)**.** *For any* $e = (K, p, g)$*, if* $\#K = 2$ *then* $\sum_{c \in K} \ell^i_c(e) = 1$*.*

The last two axioms reflect two conflicting approaches to the concept of electoral legitimacy. Under the first "absolutist" approach, a candidate's legitimacy depends only on her absolute support, both in terms of popular and electoral vote. That is, candidate c's legitimacy depends only on p_c and g_c, and is independent of the electoral record of other candidates in the election. In particular, the difficulty of achieving legitimacy because of the presence of a strong third candidate does not influence the requirements for being legitimate. This absolutist approach to electoral legitimacy is embodied in the following Independence axiom.

Definition 5 (Independence)**.** *For any* $e = (K, p, g)$*,* $e' = (K', p', g')$*, and any* $c \in K \cap K'$*, if* $p_c = p'_c$ *and* $g_c = g'_c$*, then* $\ell^i_c(e') = \ell^i_c(e)$*.*

Independence is consistent with Bob Dole's questioning of B. Clinton's legitimacy after the 1992 election, in spite of B. Clinton winning the popular vote (B. Clinton secured 43% of the popular, versus 37.4% for H.W. Bush, and 18.9% for the independent candidate R. Perot). Dole's reluctance to acknowledge B. Clinton's legitimacy reveals a disregard for the relative popular vote as a source of electoral legitimacy, and a focus on the *absolute* popular vote.

In contrast, under the "relativist" approach, the legitimacy of a candidate in a given election depends on the level of her competition, i.e., the number and strength of the independent candidates. In order to cancel out the effect of electoral competition, Relativity requires legitimacy judgements for the two top candidates to coincide for the election and the reduced election obtained after removing the independent candidates.

Definition 6 (Relativity)**.** *For any* $e = (K, p, g)$*,* $\ell^i_d(e) = \ell^i_d(\tilde{e})$ *and* $\ell^i_r(e) = \ell^i_r(\tilde{e})$*.*

As a result, legitimacy judgements for the two main candidates depend only on their *relative* shares of popular and electoral vote. Relativity allows a candidate to be deemed legitimate even if she did not secure an absolute majority of either of the two votes (but secured more of the popular or electoral vote than her opponents). This type of judgments is consistent with critiques of the 2000 and 2016 U.S. election emphasizing A. Gore's and H. Clinton's win of the popular vote. Although neither A. Gore nor H. Clinton secured an absolute majority of the popular vote (respectively, 48.4% and 48.1%), them winning more popular votes than any other candidates in the election was viewed by supporters as a source of political legitimacy. This reveals a disregard for the absolute popular vote and a focus on the relative popular vote.

3.2 Characterizations

In this section, we show that the above restrictions force legitimacy judgment to be based on *linear* trade-offs between either the absolute or the relative popular and electoral votes. We say that a legitimacy function is **absolute linear** if (i) it is linear with respect to the *absolute* shares of the popular and electoral votes p_c and g_c, and (ii) such that any candidate who gets a majority (minority) of both the popular and the electoral vote is considered legitimate (illegitimate). Formally, there exists $\alpha \in \mathbb{R}_+$ such that for any $e = (K, p, g)$ and any $c \in K$,

$$\left[p_c - \frac{1}{2}\right] + \alpha \left[g_c - \frac{1}{2}\right] > 0 \quad \text{implies} \quad \ell_c^i(e) = 1,$$
$$\left[p_c - \frac{1}{2}\right] + \alpha \left[g_c - \frac{1}{2}\right] < 0 \quad \text{implies} \quad \ell_c^i(e) = 0.$$

An example of an absolute linear legitimacy function is illustrated in Figure 2(a). Our four first restrictions together with Independence characterize the absolute linear legitimacy functions.

Theorem 1. *A legitimacy function satisfies Monotonicity, Convexity, Anonymity, Resoluteness and Independence if and only if it is absolute linear.*

We say that a legitimacy function is **relative linear** if (i) it is linear with respect to the *relative* shares of the popular and electoral votes $\hat{p}_c$ and $\hat{g}_c$ and (ii) such that any candidate who wins (loses) both the popular and the electoral vote is considered legitimate (illegitimate). That is, there exists $\alpha \in \mathbb{R}_+$ such that for any $e = (K, p, g)$ and any $c \in K$,

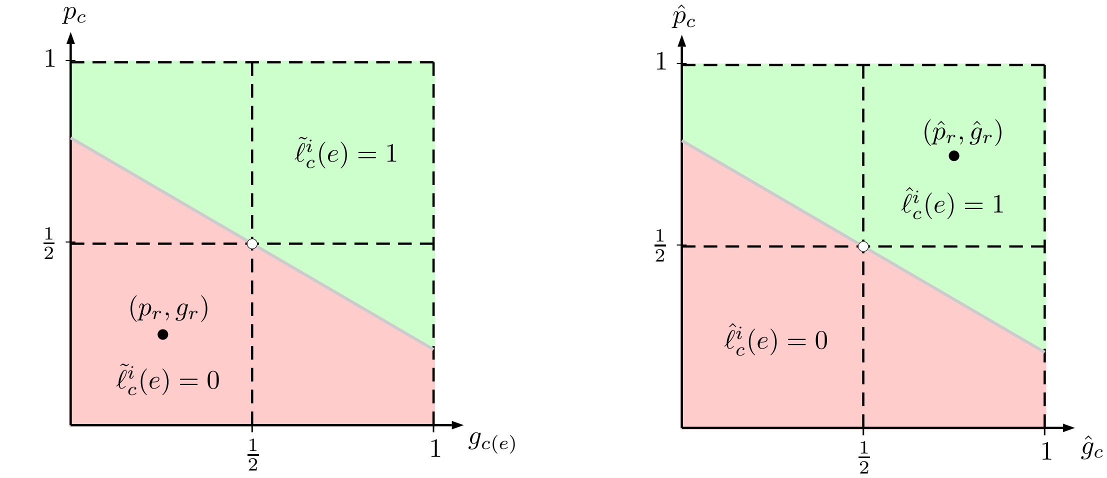
$$\left[\hat{p}_c - \frac{1}{2}\right] + \alpha \left[\hat{g}_c - \frac{1}{2}\right] > 0 \quad \text{implies} \quad \ell_c^i(e) = 1,$$
$$\left[\hat{p}_c - \frac{1}{2}\right] + \alpha \left[\hat{g}_c - \frac{1}{2}\right] < 0 \quad \text{implies} \quad \ell_c^i(e) = 0.$$

An example of an absolute linear legitimacy function is illustrated in Figure 2(b).[13] Our four first restrictions together with Relativity characterize the relative linear legitimacy functions.

Theorem 2. *A legitimacy function satisfies Monotonicity, Convexity, Anonymity, Resoluteness and Relativity if and only if it is relative linear.*

Observe that when the legitimacy function is relative linear, one and only one of the two top candidates is legitimate (while the other candidates are illegitimate). In contrast,

[13] In both the absolute and the relative case, the value of the legitimacy function at the border between the legitimate and the illegitimate area is not specified by Theorems 1 and 2. The border must, however, satisfy Convexity, Anonymity, and Resoluteness. This implies that the border is the juxtaposition of a "legitimate" and an "illegitimate" linear segments with the same slope (one of the two segments could have length zero).



(a) An absolute linear legitimacy function $\tilde{\ell}$.

(b) A relative linear legitimacy function $\hat{\ell}$.

Figure 2

if the legitimacy function is absolute linear, it can be that none of the candidates are legitimate. Observe also that, even when an absolute and a relative linear legitimacy function share the same slope parameter α, the two functions do not agree on all legitimacy judgments. In fact, some legitimacy judgments *systematically* differ between relative and absolute linear legitimacy functions, regardless of their slopes. For example, consider the electoral record e with $(p_r, g_r) = (1/4, 1/4)$, $(p_d, g_d) = (1/8, 1/8)$, and $(p_c, g_c) = (1/16, 1/16)$ for all $c \in K \backslash \{r, d\}$. As illustrated in Figure 2, $(\hat{p}_r, \hat{g}_r) = (3/4, 3/4)$, and r is therefore legitimate according to any relative linear legitimacy function. Differently, because $(p_r, g_r) = (1/4, 1/4) << (1/2, 1/2)$, r is illegitimate according to any absolute linear legitimacy function.

4 Electoral legitimacy partial orders

Convexity, Monotonicity, Anonymity, and Resoluteness are mild restrictions. It is therefore reasonable to assume that observers in N have either absolute or relative *linear* legitimacy functions. Observers may, however, disagree on whether legitimacy should be relative or absolute. They could also disagree on the way the (absolute or relative) share of the popular vote should be weighted against the (absolute or relative) share of the electoral vote, as reflected in the slope of linear legitimacy functions.

Assuming linear legitimacy functions, the population of observers can be described by a distribution $(t^i, \alpha^i)_{i \in N}$ where t_i is observer i's type (absolute or relative), and α_i is the slope of i's linear legitimacy function. In practice, it can be hard to elicit electoral legitimacy judgments and empirically estimate $(t^i, \alpha^i)_{i \in N}$. Even with carefully crafted survey questions, some respondents may not shield their responses from personal opinions on the candidates and answers are bound to capture some noise from non-electoral sources of legitimacy (in particular, survey answers are likely to violate Anonymity). As we now show, it is however possible to reach politically relevant legitimacy comparisons without actually estimating $(t^i, \alpha^i)_{i\in N}$, and without imposing further restrictions on the form of the legitimacy functions.

Consider two elections $e = (K, p, g)$ and $e' = (K', p', g')$, and two candidates $c \in K$ and $c' \in K'$. Suppose that for a particular distribution of legitimacy functions $\ell \coloneqq (\ell^i)_{i\in N}$, there are more observers who believe that c is legitimate in e than there are observers who believe that c' is legitimate in e'. Then it seems natural to conclude that, given ℓ, candidate c is more legitimate in election e than c' was in election e'.

Of course, the situation we just described is dependent on ℓ. Because ℓ is typically unknown and can be hard to observe, it would be useful to reach *robust* conclusions on the relative legitimacy of c and c', that is, conclusions that are to some extent *in*dependent of ℓ. Suppose that all legitimacy functions ℓ^i belong to some domain $\mathcal{F}$ and that

$$\{\ell^i \in \mathcal{F} \mid \ell^i_c = 1\} \supseteq \{\ell^i \in \mathcal{F} \mid \ell^i_{c'}(e') = 1\}. \tag{1}$$

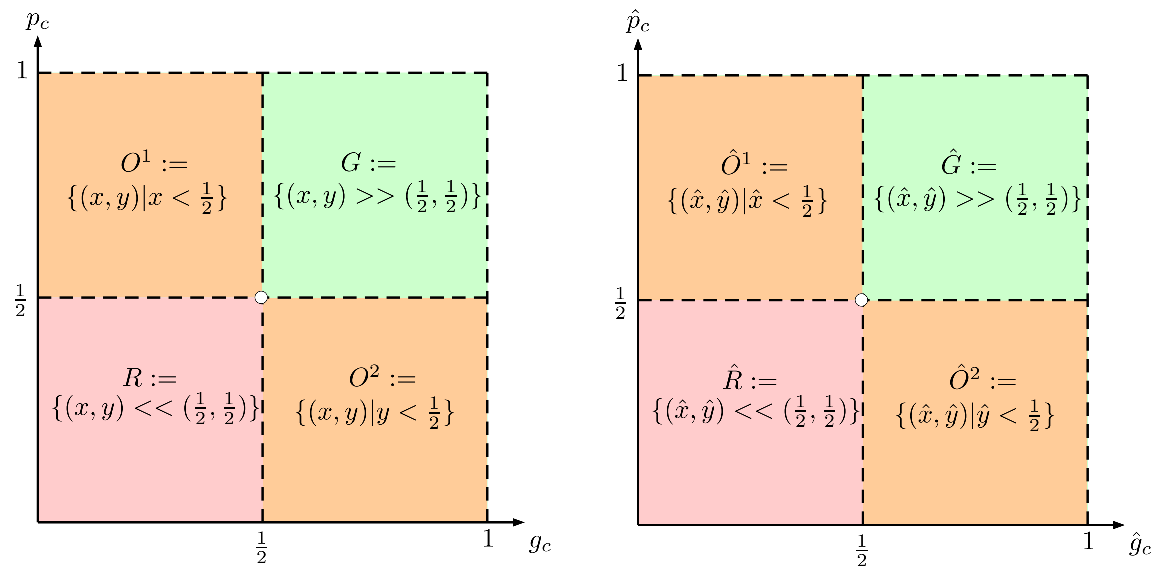
Then for any distribution ℓ with $\ell^i \in \mathcal{F}$ for all $i \in N$, *at least as many* observers believe that c is legitimate in e than there are observers who believe that c' is legitimate in e'. In this case, we can robustly conclude that, given domain $\mathcal{F}$, c' is no more legitimate in e' than c is in e, which we denote $(c; e) \succeq^{\mathcal{F}} (c'; e')$. In this way, any domain $\mathcal{F}$ defines a legitimacy partial order over pairs $(c; e)$.

If in addition, the inclusion in (1) is strict (i.e., $\supseteq$ is replaced by $\supset$), then there exists distributions ℓ with $\ell^i \in \mathcal{F}$ for all $i \in N$ for which *more* observers believe that c is legitimate in e than there are observers who believe that c' is legitimate in e'. In this case, we say that, given domain $\mathcal{F}$, c is more legitimate in e than c' is in e', which we denote $(c; e) \succ^{\mathcal{F}} (c'; e')$.

Henceforth, our focus will be on partial orders $\succeq^{\mathcal{L}}, \succeq^{\mathcal{A}}$ and $\succeq^{\mathcal{R}}$, where $\mathcal{A}$ is the domain of *absolute* linear legitimacy functions, $\mathcal{R}$ the domain of *relative* linear legitimacy functions, and $\mathcal{L} = \mathcal{A} \cup \mathcal{R}$ the domain of linear legitimacy functions. For brevity, we only describe $\succeq^{\mathcal{A}}$. Partial order $\succeq^{\mathcal{R}}$ is identical to $\succeq^{\mathcal{A}}$ with p and g replaced by $\hat{p}$ and $\hat{g}$ (and is illustrated in Figures 3(b) and 4(b)), and $\succeq^{\mathcal{L}}$ follows directly from $\succeq^{\mathcal{A}}$ and $\succeq^{\mathcal{R}}$.

Comparisons in terms of $\succeq^{\mathcal{A}}$ are easily described graphically. The simplest cases occur when candidates' electoral record lie in different "regions" of the $(0, 1) \times (0, 1)$ graph. This is illustrated in Figure 3(a). For example, if c secures the majority of both the electoral and the popular vote, i.e., (p_c, g_c) lies in region $G \coloneqq \{(x, y) >> (1/2, 1/2)\}$, then $(c; e) \succeq^{\mathcal{A}} (c'; e')$ for all $(c'; e')$. In addition, if c' secures the majority of only *one* of the two votes, e.g., c''s record lies in $O^1 \coloneqq \{(x, y) \mid x < 1/2\}$ or $O^2 \coloneqq \{(x, y) \mid y < 1/2\}$, then $(c; e) \succ^{\mathcal{A}} (c'; e')$.

The most interesting cases occurs when c and c' both secure a majority of one of the two votes, but not of the other. For example, c and c' could be two U.S. presidents who were elected without a majority of the popular vote. In this case, the legitimacy



(a) Slightly abusing the notation, we have $G \succ^{\mathcal{A}} O^1 \cup O^2 \cup R$, $O^1 \succ^{\mathcal{A}} R$, $O^2 \succ^{\mathcal{A}} R$.

(b) Slightly abusing the notation, we have $\hat{G} \succ^{\mathcal{A}} \hat{O}^1 \cup \hat{O}^2 \cup \hat{R}$, $\hat{O}^1 \succ^{\mathcal{A}} \hat{R}$, $\hat{O}^2 \succ^{\mathcal{A}} \hat{R}$.

Figure 3

of candidates c and c' can be compared as illustrated in Figure 4(a). Simply draw the line passing through (p_c, g_c) and the point $(1/2, 1/2)$. If $(p'_{c'}, g'_{c'})$ lies *above* this line, then any legitimacy function in $\mathcal{A}$ that views c as legitimate also views c' as legitimate. In addition, there exists legitimacy functions in $\mathcal{A}$ for which c' is legitimate whereas c is not. Consequentially, $(c'; e') \succ^{\mathcal{A}} (c; e)$. By symmetry, if $(p'_{c'}, g'_{c'})$ lies *below* the line passing through (p_c, g_c) and the point $(1/2, 1/2)$, then $(c; e) \succ^{\mathcal{A}} (c'; e')$.[14]

The legitimacy of c and c' cannot be compared according to $\succeq^{\mathcal{A}}$ if $(p_c, g_c) \in O^1$ and $(p'_{c'}, g'_{c'}) \in O^2$, or if $(p_c, g_c) \in O^2$ and $(p'_{c'}, g'_{c'}) \in O^1$ That is, $\succeq^{\mathcal{A}}$ cannot compare a candidate who secured a majority of the popular (electoral) vote but failed to secure a majority of the electoral (popular) vote, with another candidate who secured a majority of the electoral (popular) vote but failed to secure a majority of the popular (electoral) vote. For example, $\succeq^{\mathcal{A}}$ does not enable legitimacy comparisons between a president who was elected without a majority of the popular vote and a candidate who lost the election despite securing more than half of the popular vote.

The next proposition formally characterizes $\succeq^{\mathcal{A}}$.

Proposition 1. *For any $e = (K, p, g)$, $e' = (K', p', g')$, any $c \in K$ and any $c' \in K'$, $(c; e) \succeq^{\mathcal{A}} (c'; e')$ if and only if*

(a) c has a strict majority of both the popular and the electoral vote,

(b) c' has a strict minority of both the popular and the electoral vote,

[14] If vectors $(p_c - 1/2, g_c - 1/2)$ and $(p'_{c'} - 1/2, g'_{c'} - 1/2)$ are collinear, then the legitimacy of c and c' cannot be compared according to $\succeq^{\mathcal{A}}$.

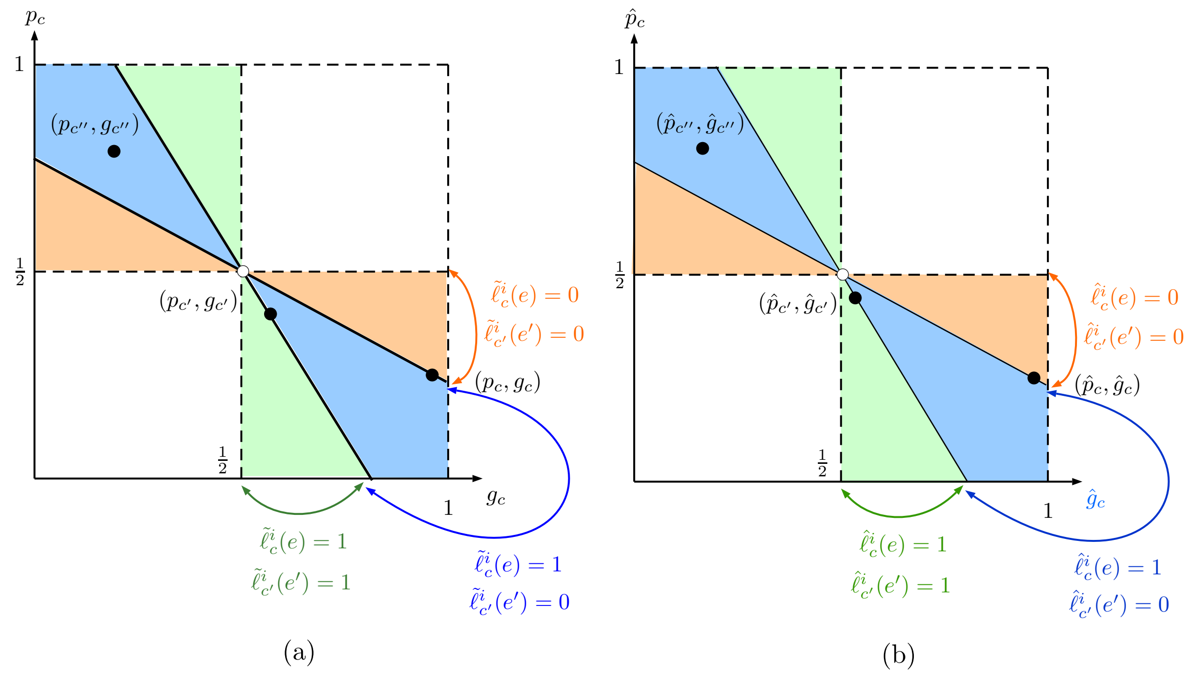


Figure 4: According to $\succeq^{\mathcal{F}}$ ($\mathcal{F} = \mathcal{A}$ for (a) and $\mathcal{F} = \mathcal{R}$ for (b)), candidate c is more legitimate in election e than is candidate c' in election e' because c's electoral record in e lies above the line passing through c''s electoral record in e' and the $(\frac{1}{2}, \frac{1}{2})$ points. Candidate c'' cannot be compared to candidates c (c') because there exists both legitimacy functions in $\mathcal{A}$ such that c (c') is legitimate and c'' is not, and legitimacy functions in $\mathcal{F}$ such that c'' is legitimate and c (c') is not.

(c) both c and c' have a strict majority of the popular vote but a strict minority of the electoral vote and

$$\frac{p'_{c'} - 1/2}{g'_{c'} - 1/2} < \frac{p_c - 1/2}{g_c - 1/2}, \text{ or}$$

(d) both c and c' have a strict majority of the electoral vote but a strict minority of the electoral vote and

$$\frac{p'_{c'} - 1/2}{g'_{c'} - 1/2} > \frac{p_c - 1/2}{g_c - 1/2}.$$

Similarly, we get the following characterization for $\succeq^{\mathcal{R}}$.

Proposition 2. *For any $e = (K, p, g)$, $e' = (K', p', g')$, any $c \in K$ and any $c' \in K'$, $(c; e) \succeq^{\mathcal{R}} (c'; e')$ if and only if*

(a) c won both the popular and the electoral vote,

(b) c' lost both the popular and the electoral vote,

(c) both c and c' won the popular vote but lost the electoral vote and

$$\frac{\hat{p}'_{c'} - 1/2}{\hat{g}'_{c'} - 1/2} < \frac{\hat{p}_c - 1/2}{\hat{g}_c - 1/2}, \text{ or}$$

(d) both c and c' won the electoral vote but lost the popular vote and

$$\frac{\hat{p}'_{c'} - 1/2}{\hat{g}'_{c'} - 1/2} > \frac{\hat{p}_c - 1/2}{\hat{g}_c - 1/2}.$$

Finally, note that $(c; e) \succeq^{\mathcal{L}} (c'; e')$ if and only if $(c; e) \succeq^{\mathcal{A}} (c'; e')$ and $(c; e) \succeq^{\mathcal{R}} (c'; e')$.

5 Applying $\succeq$ to U.S. presidential elections

We focus on contentious U.S. presidential elections between 1876 and 2016. Recall that an election is (absolute or relative) contentious if the candidate who won the election did not secure the (absolute or relative) majority of the popular vote.[15] For brevity, we write $c \succeq^{\mathcal{F}} c'$, when candidate c was more legitimate according to $\succeq^{\mathcal{F}}$ than c', omitting the reference to the elections in which c and c' ran for the presidency. To avoid confusion when a candidate participated in several elections, we add an indicator of the corresponding election year to the candidates' name (e.g., G.W. Bush00 and G.W. Bush04).

A first question is whether in contentious elections, the winner of an election is systematically more legitimate than the runner-up. As Figure 5 illustrates, in all but one of the absolute contentious elections, the winner was more legitimate than its runner-up according to $\succeq^{\mathcal{A}}$. This is because, except for R. Hayes' victory over S. Tilden in 1876, the runner-up in absolute contentious elections never secured a majority of the popular vote. In these elections, although the winners also failed to obtain a majority of the popular vote, runner-ups were prevented from securing more than half of the popular vote due to the presence of smaller candidates. Hence, the winner earning a majority of the *electoral* vote was sufficient to guarantee him a higher legitimacy than his runner-ups according to $\succeq^{\mathcal{A}}$. In particular, D. Trump $\succ^{\mathcal{A}}$ H. Clinton and G.W. Bush00 $\succ^{\mathcal{A}}$ A. Gore.

In all but four elections, the elected president won the popular vote, which can be seen in Figure 5 by observing that only four of the arrows linking elected presidents to their runner-ups points upward.[16] Because elected presidents must win an *absolute* majority of the electoral vote, elected presidents also win a *relative* majority of the electoral vote. Hence, in all but the four relative contentious elections, elected presidents were also more legitimate than their runner-ups in terms of $\succeq^{\mathcal{R}}$, and consequentially, in terms of $\succeq^{\mathcal{L}}$. In this sense, except for the four winners who did not win the popular vote, the winners of all U.S. presidential elections were uncontroversially more legitimate than their direct opponents.[17]

Comparisons are more ambiguous for the four candidates who clinched the presidency without winning the popular vote, as can be seen in Figure 6. Let us focus on the two

[15] The complete absolute and relative electoral records are represented in Figures 8 and 9 in the Appendix

[16] For a more general picture, see Figure 9 in the Appendix, where the *relative* electoral record of all but 4 winners lies in the South-East green region above the $(1/2, 1/2)$ point.

[17] The 1876 election was both relative and absolute contentious, which explains why four and not five of the elected U.S. president failed to dominate their direct opponents in terms of $\succeq^{\mathcal{L}}$.

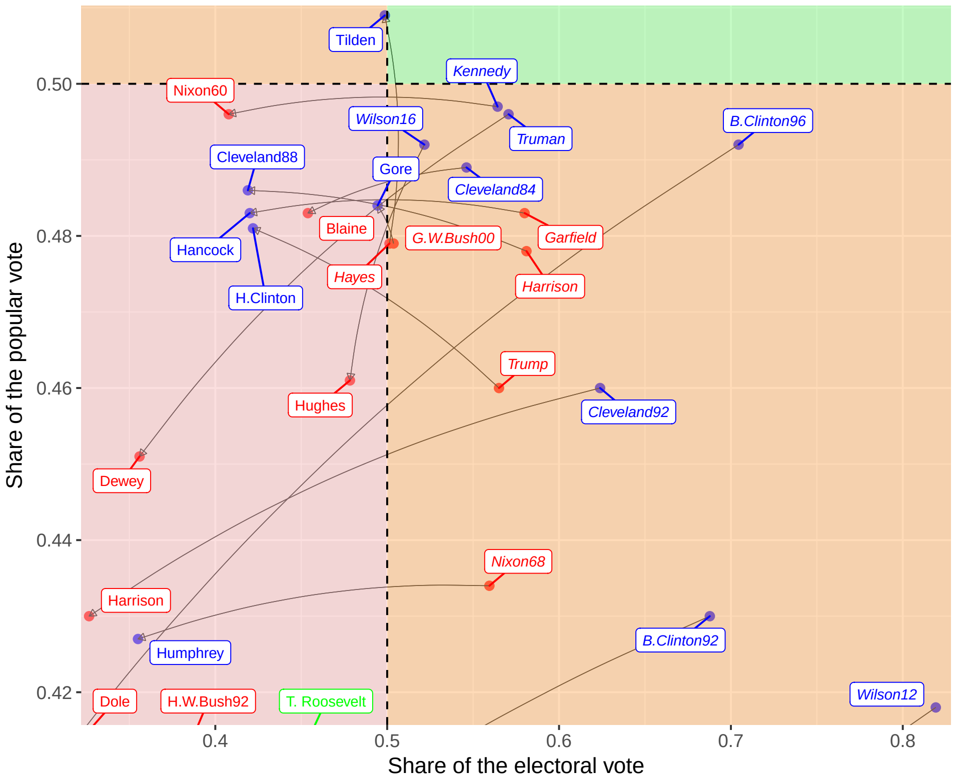


Figure 5: Absolute contentious U.S. presidential elections. In all figures representing electoral records for U.S. presidential elections, blue candidates are democrats, red candidates are republicans, and green candidates are independent.

most recent cases of G.W. Bush00 v. A. Gore and D. Trump v. H. Clinton. As we discussed, Trump and G.W. Bush00 are more legitimate than A. Gore and H. Clinton in terms of $\succeq^{\mathcal{A}}$ because neither A. Gore not H. Clinton managed to secure a majority of either the absolute popular or absolute electoral vote (whereas, by virtue of being elected, D. Trump and G.W. Bush00 secured an absolute majority of the electoral vote). However, A. Gore and H. Clinton famously won the popular vote, which implies that G.W. Bush00 and D. Trump cannot be compared with A. Gore and H. Clinton according to $\succeq^{\mathcal{R}}$ (as a consequence, comparisons in terms of $\succeq^{\mathcal{L}}$ are also impossible).

Comparing the legitimacy of G.W. Bush00 and A. Gore or D. Trump and H. Clinton requires more information on the distribution of legitimacy functions in N. Clearly, for more than half of the observers in N to believe that A. Gore or H. Clinton are more legitimate than G.W. Bush00 or D. Trump, more than half of the observers must be relativists (as we showed that D. Trump $\succ^{\mathcal{A}}$ H. Clinton and G.W. Bush00 $\succ^{\mathcal{A}}$ A. Gore). More precisely, in the case of D. Trump v. H. Clinton, more than half of the observers must be relativist *and* have a slope parameter $\alpha_i \in (0, -0.15)$ for H. Clinton to have more legitimacy support than D. Trump among N, as illustrated in Figure 6. Observers with $\alpha_i \in (0, -0.15)$ who strongly favor the relative share of the popular vote as a source

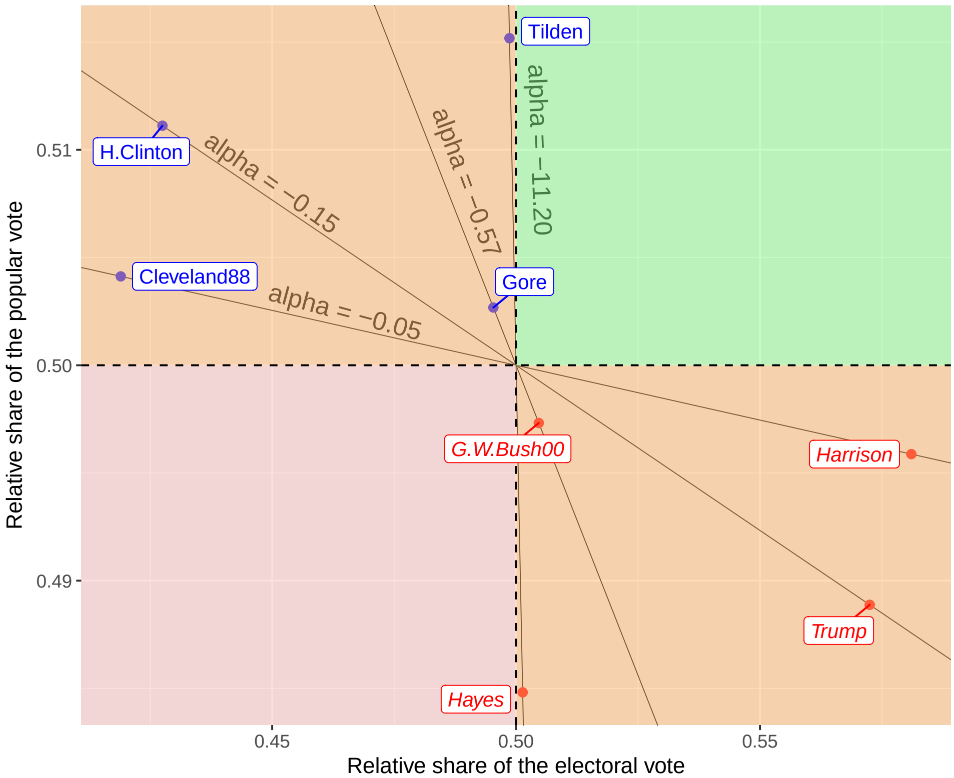


Figure 6: Relative contentious U.S. presidential elections.

of legitimacy certainly exist.[18] But they are unlikely to represent more than half of the observer's population, and one can still expect (albeit in a weaker sense than $\succ^{\mathcal{L}}$) that D. Trump's legitimacy would be supported by more observers' than H. Clinton's legitimacy. In the case of G.W. Bush00 v. A. Gore, more than half of the observers must be relativists *and* have a slope parameter $\alpha_i \in (0, -0.57)$ for A. Gore to have more legitimacy support than G.W. Bush00. This is strictly weaker than requiring that more than half of the observers be relativists *and* have a slope parameter $\alpha_i \in (0, -0.15)$. Hence, although it remains unlikely, A. Gore having more legitimacy support than G.W. Bush00 is more likely than H. Clinton having more legitimacy support than D. Trump.

Nevertheless, comparisons between winners and runner-ups in relative contentious elections remain uncertain without further knowledge of the distribution of legitimacy functions. Another question is whether in relative contentious and other elections, legitimacy can be compared *among winners*. Such comparisons rely on the technique described in Figure 4: a candidate c is more legitimate than candidate c' in terms of $\succ^{\mathcal{F}}$ if c''s electoral record lies above the line between c's electoral record and the mid point $(1/2, 1/2)$.

Comparisons in terms of $\succ^{\mathcal{R}}$ are illustrated in Figure 6 for relative contentious elections. Note that two of the six comparisons are non-trivial in the sense that they do not

[18] As a matter of fact, some believe that only the popular vote should matter, and hence have a slope parameter α arbitrarily close to zero.

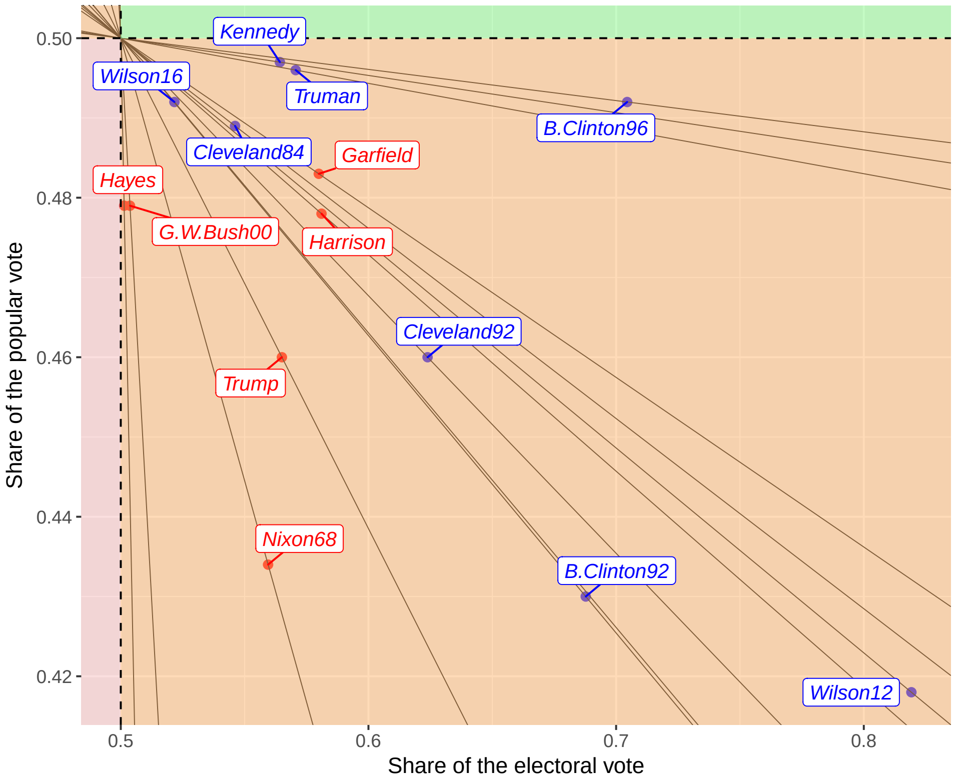


Figure 7: Comparing the legitimacy of the winners of absolute contentious U.S. presidential elections.

follow from Monotonicity alone. In particular, D. Trump $\succ^{\mathcal{R}}$ G.W. Bush00 in a non-trivial way. This comparison is consistent with the fact that a more extreme distribution of legitimacy functions (skewed toward observers who strongly favor the relative popular vote) would have been required for H. Clinton to be more legitimate than D. Trump, than for A. Gore to be more legitimate than G.W. Bush00.[19]

Comparisons in terms of $\succ^{\mathcal{A}}$ are illustrated in Figure 7 for absolute contentious elections. Again, many of the comparisons in the figure do not follow from Monotonicity alone. As with $\succ^{\mathcal{R}}$, we find that D. Trump $\succ^{\mathcal{A}}$ G.W. Bush00, which implies D. Trump $\succ^{\mathcal{L}}$ G.W. Bush00. In this sense, D. Trump's election in 2016 was more legitimate than G.W. Bush's election in 2000.

However, according to $\succeq^{\mathcal{A}}$, only three absolute contentious elections saw the winner clinch the presidency with a less legitimate election record than that of D. Trump (namely, R. Hayes, R. Nixon68, and G.W. Bush00). Because non-contentious winner are automatically more legitimate than D. Trump in terms of $\succeq^{\mathcal{A}}$, we find that X $\succeq^{\mathcal{A}}$ D. Trump for *every* elected U.S. president X other than R. Hayes, R. Nixon68, and G.W. Bush00.

A similar result is found for $\succeq^{\mathcal{R}}$. For *every* elected U.S. president X other than R.

[19] Any comparison among winners matches with a comparison among runner-ups. For example, D. Trump $\succ^{\mathcal{R}}$ G.W. Bush00 implies A. Gore $\succ^{\mathcal{R}}$ H. Clinton.

Hayes and G.W. Bush00, X $\succ^{\mathcal{R}}$ D. Trump. Overall, X $\succ^{\mathcal{L}}$ D. Trump for *every* elected U.S. president X other than R. Hayes, R. Nixon68, and G.W. Bush00.[20] In this sense, D. Trump's election remains one of the most contentious in history in terms of electoral legitimacy.

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[20] D. Trump $\succ^{\mathcal{A}}$ {R. Hayes, G.W. Bush00}, and D. Trump $\succ^{\mathcal{R}}$ {R. Hayes, G.W. Bush00}, which implies D. Trump $\succ^{\mathcal{L}}$ {R. Hayes, G.W. Bush00}. In contrast, D. Trump $\succ^{\mathcal{A}}$ R. Nixon68, whereas R. Nixon68 $\succ^{\mathcal{R}}$ D. Trump, so R. Nixon68 and D. Trump cannot be compared in terms of $\succ^{\mathcal{L}}$.

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Appendix

A Proofs

For any $c \in \mathbb{K}$, let $\mathcal{E}(c)$ denote the subset of elections where c is among the candidates. For any $S \subset \mathbb{K}$, let $\mathcal{E}(S)$ denote the subset of elections where the set of candidates coincides with S. For any $c, c' \in \mathbb{K}$, let $\mathcal{E}_{c,c'}$ denote the subset of elections where candidates c and c' are the two top candidates.

A.1 Theorem 1

Proof. Take any $c, c' \in \mathbb{K}$.

Step 1: ℓ_c^i is absolute linear on $\mathcal{E}_{c,c'}$.

By Convexity, the set $\mathcal{E}^1(\{c, c'\}) = \{e \in \mathcal{E}(\{c, c'\}) \mid \ell_c^i(e) = 1\}$ is convex. By Anonymity and Resoluteness, this implies that the set $\mathcal{E}^0(\{c, c'\}) = \{e \in \mathcal{E}(\{c, c'\}) \mid \ell_c^i(e) = 0\}$ is also convex. By Resoluteness, ℓ_c^i is single-valued over $\mathcal{E}(\{c, c'\})$, so these two sets are disjoint. Hence, by the Separating Hyperplane Theorem, there exists $\alpha \in \mathbb{R}_+$ and $\beta \in [0, 1]$ such that for any $e \in \mathcal{E}(\{c, c'\})$, $p_c + \alpha g_c > \beta(1 + \alpha)$ implies $\ell_c^i(e) = 1$ and $p_c + \alpha g_c < \beta(1 + \alpha)$ implies $\ell_c^i(e) = 0$.

Now, suppose that $\beta \neq 1/2$. Let $\overrightarrow{1/2} := (1/2, 1/2)$. Then either (i) there exists $e = (\{c, c'\}, p, g) \in \mathcal{E}(\{c, c'\})$ such that $(p_c, g_c) << \overrightarrow{1/2}$ and $\ell_c^i(e) = 1$, or (ii) there exists $e' = (\{c, c'\}, p', g') \in \mathcal{E}(\{c, c'\})$ such that $(p'_c, g'_c) >> \overrightarrow{1/2}$ such that $\ell_c^i(e') = 0$. Let $sym(e) := (\{c, c'\}, (1, 1) - p, (1, 1) - g)$ and $sym(e') := (\{c, c'\}, (1, 1) - p', (1, 1) - g')$, the images of e and e' by central symmetry around $\overrightarrow{1/2}$. By Anonymity and Resoluteness, we have (i') $\ell_c^i(sym(e)) = 0$, and (ii') $\ell_c^i(sym(e')) = 1$. But then, in either (i) or (ii), there exists elections $\underline{e} = ((\{c, c'\}, \underline{p}, \underline{g})$ and $\bar{e} = ((\{c, c'\}, \bar{p}, \bar{g})$ satisfying $(\underline{p}_c, \underline{g}_c) << (\bar{p}_c, \bar{g}_c)$ with $\ell^i_c(\underline{e}) = 1$ and $\ell^i_c(\bar{e}) = 0$, contradicting Monotonicity. Hence, $\beta = 1/2$.

Step 2: ℓ^i_c is absolute linear in every election where c is a candidate.

Take any $e = (K, p, g) \in \mathcal{E}(c)$. Define $\tilde{e} = (\{c, c'\}, \tilde{p}, \tilde{g})$, where $\tilde{p}_c = p_c$ and $\tilde{g}_c = g_c$. By Step 1, $\tilde{p}_c + \alpha\tilde{g}_c > (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 1$ and $\tilde{p}_c + \alpha\tilde{g}_c < (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 0$. By Independence, since $p_c = \tilde{p}_c$ and $g_c = \tilde{g}_c$, $p_c + \alpha g_c > (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 1$ and $p_c + \alpha g_c < (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 0$.

Step 3: the functions ℓ^i_c have the same slope for each candidate c.

Take any $e = (K, p, g)$ and $c'' \in K$. If $c = c''$, it obvious by Step 2 that $\ell^i_c = \ell^i_{c''}$. So assume $c \neq c''$. There are two cases.

First Case: $c \notin K$. Consider $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ such that (i) $\tilde{K} = K \backslash \{c''\} \cup \{c\}$, (ii) $\tilde{p}_t = p_t$ and $\tilde{g}_t = g_t$ for all $t \in K\backslash\{c''\}$, and (iii) $\tilde{p}_c = p_{c''}$ and $\tilde{g}_c = g_{c''}$. By Step 2, since $\tilde{e} \in \mathcal{E}(c)$, $\tilde{p}_c + \alpha\tilde{g}_c > (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 1$ and $\tilde{p}_c + \alpha\tilde{g}_c < (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 0$. By Anonymity, $\ell^i_{c''}(e) = \ell^i_c(\tilde{e})$, so $p_{c''} + \alpha g_{c''} > (1/2)(1+\alpha)$ implies $\ell^i_{c''}(e) = 1$ and $p_{c''} + \alpha g_{c''} < (1/2)(1+\alpha)$ implies $\ell^i_{c''}(e) = 0$.

Second Case: $c \in K$. Then, consider instead $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ such that (i) $\tilde{K} = K$, (ii) $\tilde{p}_t = p_t$ and $\tilde{g}_t = g_t$ for all $t \in K\backslash\{c'', c\}$, and (iii) $\tilde{p}_{c''} = p_c$, $\tilde{g}_{c''} = g_c$, $\tilde{p}_c = p_{c''}$, and $\tilde{g}_c = g_{c''}$. By Step 2, since $\tilde{e} \in \mathcal{E}(c)$, $\tilde{p}_c + \alpha\tilde{g}_c > (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 1$ and $\tilde{p}_c + \alpha\tilde{g}_c < (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 0$. By Anonymity, $\ell^i_{c''}(e) = \ell^i_c(\tilde{e})$, so $p_{c''} + \alpha g_{c''} > (1/2)(1+\alpha)$ implies $\ell^i_{c''}(e) = 1$ and $p_{c''} + \alpha g_{c''} < (1/2)(1+\alpha)$ implies $\ell^i_{c''}(e) = 0$.

□

A.2 Theorem 2

Proof. Take $c', c'' \in \mathbb{K}$.

Step 1: ℓ^i is relative linear on $\mathcal{E}_{c',c''}$.

By Step 1 in the proof of Theorem 1, we know that Monotonicity, Convexity, Anonymity and Resoluteness imply there exists $\alpha > 0$ such that for any $e = (\{c', c''\}, p, g) \in \mathcal{E}(\{c', c''\})$, and any $c \in \{c', c''\}$, $p_c + \alpha g_c > (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 1$ and $p_c + \alpha g_c < (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 0$. By Relativity, it thus follows that for any $e = (K, p, g) \in \mathcal{E}_{c',c''}$, and any $c \in \{c', c''\}$, $\hat{p}_c + \alpha\hat{g}_c > (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 1$ and $\hat{p}_c + \alpha\hat{g}_c < (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 0$. By Resoluteness, either $\ell^i_{c'}(e) = 0$ or $\ell^i_{c''}(e) = 0$. Since c' and c'' are the two top candidates in election e (by definition of $\mathcal{E}_{c',c''}$). Monotonicity implies that $\ell^i_c(e) = 0$ for any $c \in K\backslash\{c', c''\}$. We conclude that there exists $\alpha > 0$ such that for any $e = (K, p, g) \in \mathcal{E}_{c',c''}$, and any $c \in K$, $\hat{p}_c + \alpha\hat{g}_c > (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 1$ and $\hat{p}_c + \alpha\hat{g}_c < (1/2)(1+\alpha)$ implies $\ell^i_c(e) = 0$.

Step 2: ℓ^i is relative linear.

Take any $e = (K, p, g)$. Define $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ with $\tilde{K} = K \backslash \{d, r\} \cup \{c', c''\}$, $\tilde{p}_{c'} = p_d$, $\tilde{p}_{c''} = p_r$, $\tilde{g}_{c'} = g_d$, $\tilde{g}_{c''} = g_r$, $\tilde{p}_c = p_c$ for any $c \in K\backslash\{d, r\}$ and $\tilde{g}_c = g_c$ for any $c \in K\backslash\{d, r\}$. We have $\tilde{e} \in \mathcal{E}_{c',c''}$. By Step 1, for any $c \in \tilde{K}$, $\hat{\tilde{p}}_c(\tilde{e})+\alpha\hat{\tilde{g}}_c(\tilde{e}) > (1/2)(1+\alpha)$ implies $\ell^i_c(\tilde{e}) = 1$ and $\hat{\tilde{p}}_c(\tilde{e}) + \alpha\hat{\tilde{g}}_c(\tilde{e}) < (1/2)(1 + \alpha)$ implies $\ell^i_c(\tilde{e}) = 0$. By Anonymity, since $\ell^i_c(\tilde{e}) = \ell^i_c(e)$ for any $c \in K\backslash\{d, r\}$, $\ell^i_{c'}(\tilde{e}) = \ell^i_d(e)$ and $\ell^i_{c''}(\tilde{e}) = \ell^i_r(e)$, we conclude that for any $c \in K$, $\hat{p}_c + \alpha\hat{g}_c > (1/2)(1 + \alpha)$ implies $\ell^i_c(e) = 1$ and $\hat{p}_c + \alpha\hat{g}_c < (1/2)(1 + \alpha)$ implies $\ell^i_c(e) = 0$. $\square$

A.3 Proposition 1

Proof. Let $e = (K, p, g)$, $e' = (K', p', g')$, $c \in K$ and $c' \in K'$. We only treat two cases. All other cases follow in a similar fashion (or from monotonicity alone).

First Case: $p_c > 1/2 > g_c$ and $g'_{c'} > 1/2 > p'_{c'}$. We get three subcases.

A) Assume $(p_c - 1/2)(g'_{c'} - 1/2) > (1/2 - p'_{c'})(1/2 - g_c)$. Then, let α, α' such that

$$\alpha = \frac{p_c - 1/2}{1/2 - g_c} \quad \text{and} \quad \hat{\alpha} = \frac{1/2 - p'_{c'}}{g'_{c'} - 1/2}$$

Let $\ell \in \mathcal{A}$ such that $\ell_c = 0$ and for any $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ and any $\tilde{c} \in \tilde{K}$, $\tilde{p}_{\tilde{c}}+\alpha\tilde{g}_{\tilde{c}} > (1/2)(1+\alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 1$, and $\tilde{p}_{\tilde{c}} + \alpha g_{\tilde{c}} < (1/2)(1 + \alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 0$.

Similarly, let $\hat{\ell} \in \mathcal{A}$ such that $\hat{\ell}_{c'}(e') = 0$ and for any $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ and any $\tilde{c} \in \tilde{K}$, $\tilde{p}_{\tilde{c}} + \hat{\alpha}\tilde{g}_{\tilde{c}} > (1/2)(1 + \hat{\alpha})$ implies $\hat{\ell}_{\tilde{c}}(\tilde{e}) = 1$, and $\tilde{p}_{\tilde{c}} + \hat{\alpha}g_c < (1/2)(1 + \hat{\alpha})$ implies $\hat{\ell}_{\tilde{c}}(\tilde{e}) = 0$.

By definition of α and α', $\ell_{c'}(e') = 1$ and $\hat{\ell}_c(e) = 1$. Therefore, $(c; e)$ and $(c'; e')$ cannot be ordered by inclusion.

B) Assume $(p_c - 1/2)(g'_{c'} - 1/2) < (1/2 - p'_{c'})(1/2 - g_c)$. Define $\alpha, \hat{\alpha}$ as before and consider the same legitimacy functions $\ell, \hat{\ell} \in \mathcal{A}$ where $\ell_c = 1$ (instead of $\ell_c = 0$) and $\hat{\ell}_{c'}(e') = 1$ (instead of $\hat{\ell}_{c'}(e') = 0$). Then, $\ell_{c'}(e') = 0$ and $\hat{\ell}_c(e) = 0$. Therefore, $(c; e)$ and $(c'; e')$ cannot be ordered by inclusion.

C) Assume $(p_c - 1/2)(g'_{c'} - 1/2) = (1/2 - p'_{c'})(1/2 - g_c)$. Define $\alpha, \hat{\alpha}$ as before (now $\alpha = \hat{\alpha}$) and consider the same legitimacy functions $\ell, \hat{\ell} \in \mathcal{A}$ where $\ell_c = 1$, $\ell_{c'}(e') = 0$, $\hat{\ell}_{c'}(e') = 1$ and $\hat{\ell}_c(e) = 0$ (note that these functions are indeed in $\mathcal{A}$). We conclude that $(c; e)$ and $(c'; e')$ cannot be ordered by inclusion.

Second Case: $p_c, p'_{c'} > 1/2 > g_c, g'_{c'}$. We again get three subcases.

A) Assume $(p'_{c'}-1/2)(1/2-g_c) > (p_c-1/2)(1/2-g'_{c'})$. Let ℓ be any legitimacy function in $\mathcal{A}$ such that $\ell_c = 1$. By definition of $\mathcal{A}$, there exists $\alpha > 0$ such that for any $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ and any $\tilde{c} \in \tilde{K}$, $\tilde{p}_{\tilde{c}} + \alpha\tilde{g}_{\tilde{c}} > (1/2)(1 + \alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 1$ and $\tilde{p}_{\tilde{c}} + \alpha\tilde{g}_{\tilde{c}} < (1/2)(1 + \alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 0$. Since $\ell_c = 1$, we get $p'_{c'} + \alpha g'_{c'} > p_c + \alpha g_c \geq (1/2)(1 + \alpha)$ so that $\ell_{c'}(e') = 1$. We conclude that $(c'; e') \succeq (c; e)$.

B) Assume $(p'_{c'} - 1/2)(1/2 - g_c) > (p_c - 1/2)(1/2 - g'_{c'})$. Follows in the same fashion that $(c; e) \succeq (c'; e')$.

C) Assume $(p'_{c'} - 1/2)(1/2 - g_c) = (p_c - 1/2)(1/2 - g'_{c'})$. Then, let

$$\alpha = \frac{p_c - 1/2}{1/2 - g_c}$$

Let $\ell \in \mathcal{A}$ such that $\ell_c = 0$, $\ell_{c'}(e') = 1$, and for any $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ and any $\tilde{c} \in \tilde{K}$, $\tilde{p}_{\tilde{c}} + \alpha \tilde{g}_{\tilde{c}} > (1/2)(1 + \alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 1$, and $\tilde{p}_{\tilde{c}} + \alpha \tilde{g}_{\tilde{c}} < (1/2)(1 + \alpha)$ implies $\ell_{\tilde{c}}(\tilde{e}) = 0$.

Similarly, let $\ell' \in \mathcal{A}$ such that $\ell'_c = 0$, $\ell'_{c'}(e') = 1$, and for any $\tilde{e} = (\tilde{K}, \tilde{p}, \tilde{g})$ and any $\tilde{c} \in \tilde{K}$, $\tilde{p}_{\tilde{c}} + \alpha \tilde{g}_{\tilde{c}} > (1/2)(1 + \alpha)$ implies $\ell'_{\tilde{c}}(\tilde{e}) = 1$, and $p_c + \alpha g_c < (1/2)(1 + \alpha)$ implies $\ell'^{\tilde{c}}(\tilde{e}) = 0$.

We conclude that $(c; e)$ and $(c'; e')$ cannot be compared. □

A.4 Proposition 2

Proof. Follows similarly to Proposition 1. □

B Complete absolute and relative electoral records from 1876 to 2016

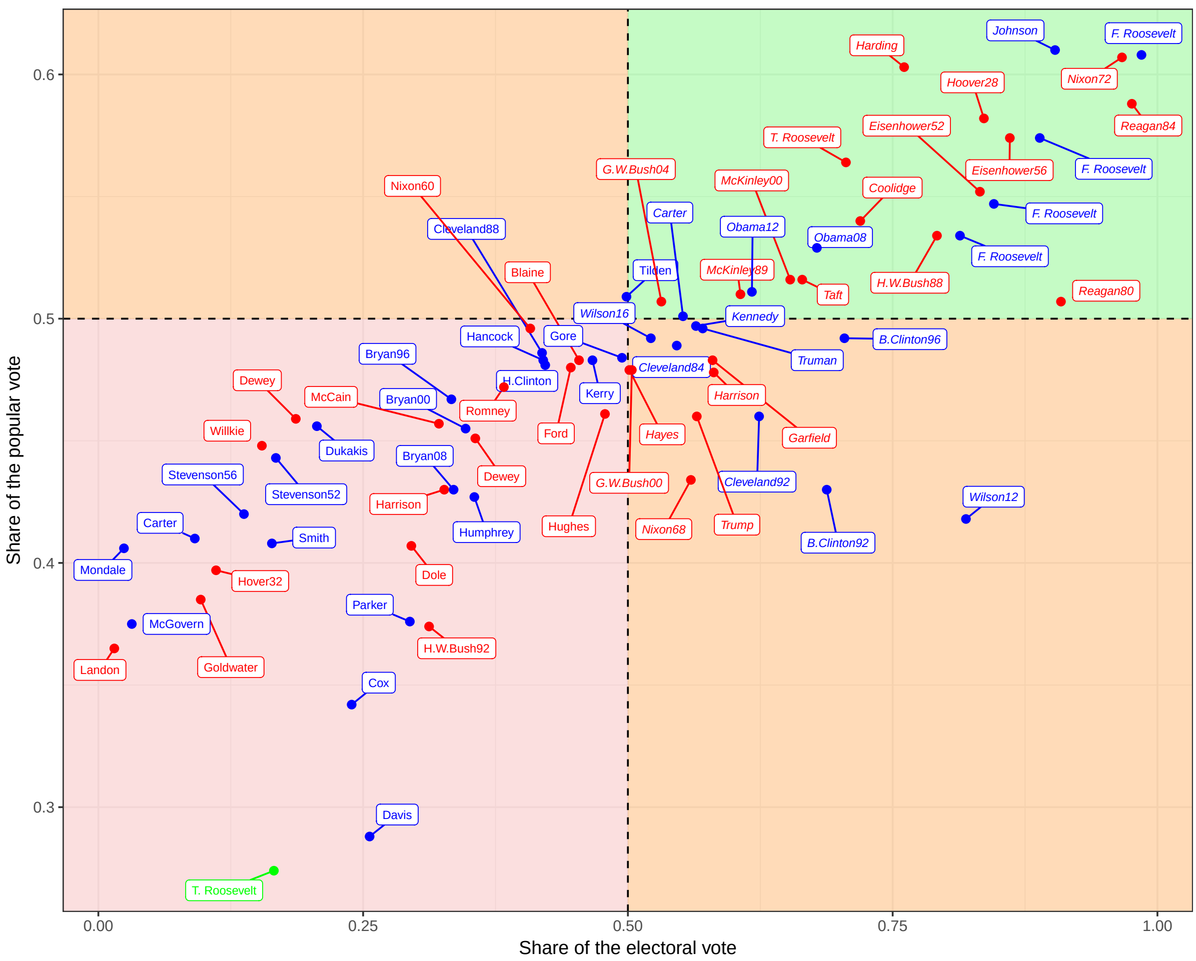
Share of the popular vote
Share of the electoral vote
0.00
0.25
0.50
0.75
1.00
0.3
0.4
0.5
0.6
Johnson
F. Roosevelt
Harding
Hoover28
Nixon72
Reagan84
Eisenhower52
T. Roosevelt
G.W.Bush04
McKinley00
Coolidge
Eisenhower56
F. Roosevelt
Nixon60
Carter
Obama12
Obama08
F. Roosevelt
Cleveland88
Blaine
Tilden
McKinley89
H.W.Bush88
Taft
Reagan80
Wilson16
Kennedy
Hancock
Gore
B.Clinton96
Bryan96
Truman
Cleveland84
H.Clinton
Dewey
McCain
Bryan00
Kerry
Harrison
Romney
Willkie
Dukakis
Ford
Hayes
Garfield
Bryan08
Stevenson56
Dewey
G.W.Bush00
Cleveland92
Wilson12
Stevenson52
Harrison
Carter
Humphrey
Hughes
Nixon68
Trump
Smith
B.Clinton92
Mondale
Hover32
Dole
Parker
McGovern
H.W.Bush92
Landon
Goldwater
Cox
Davis
T. Roosevelt

Figure 8

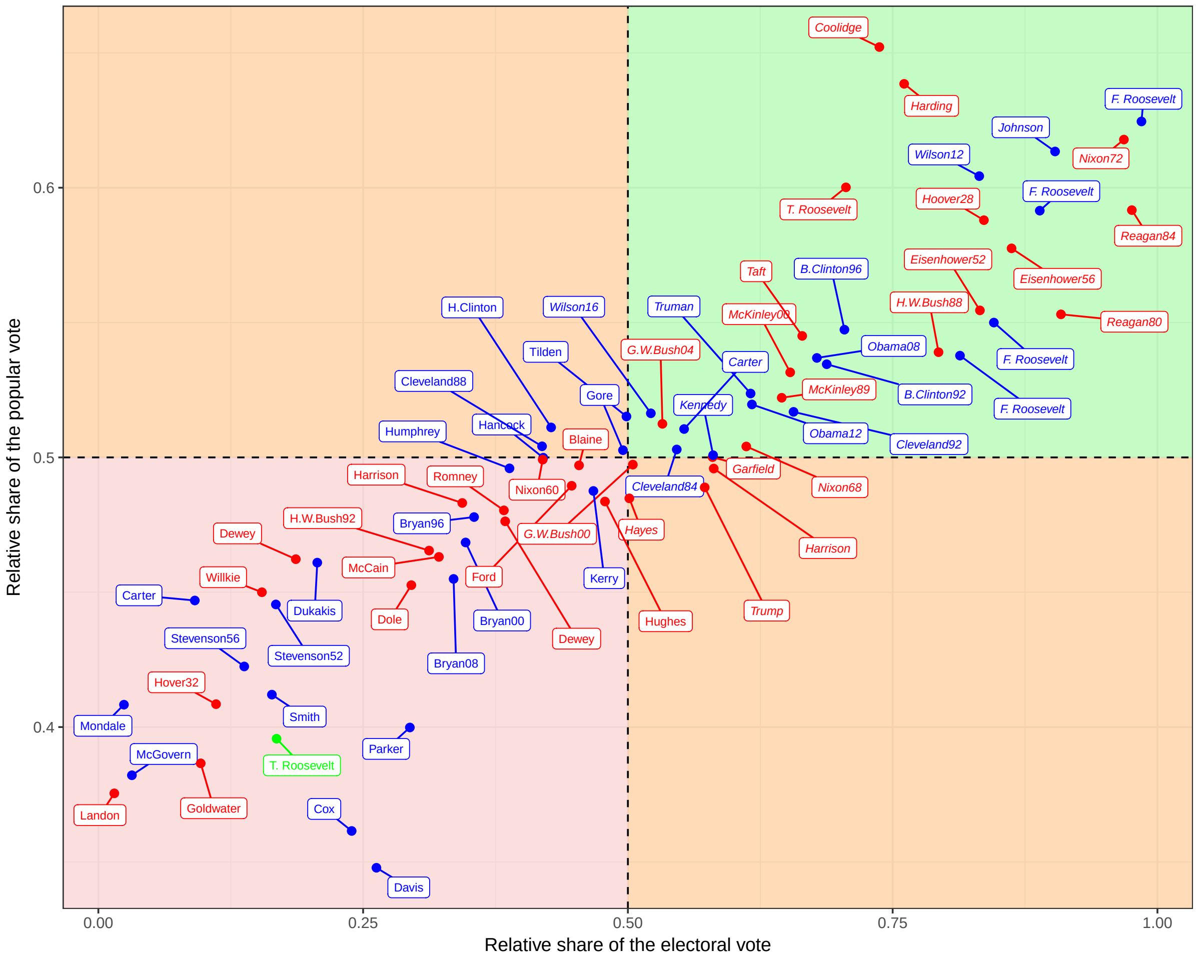
Relative share of the popular vote
Relative share of the electoral vote
0.4
0.5
0.6
0.00
0.25
0.50
0.75
1.00
Coolidge
Harding
F. Roosevelt
Johnson
Nixon72
Wilson12
T. Roosevelt
Hoover28
F. Roosevelt
Reagan84
Eisenhower52
Eisenhower56
Taft
B.Clinton96
H.W.Bush88
Reagan80
H.Clinton
Wilson16
Truman
McKinley00
Tilden
G.W.Bush04
Carter
Obama08
F. Roosevelt
Cleveland88
Gore
McKinley89
B.Clinton92
Kennedy
F. Roosevelt
Humphrey
Hancock
Obama12
Cleveland92
Blaine
Garfield
Harrison
Romney
Nixon60
Cleveland84
Nixon68
H.W.Bush92
Bryan96
G.W.Bush00
Hayes
Dewey
Harrison
McCain
Ford
Willkie
Kerry
Carter
Dukakis
Dole
Bryan00
Trump
Hughes
Dewey
Stevenson56
Stevenson52
Bryan08
Hover32
Smith
Mondale
Parker
McGovern
T. Roosevelt
Goldwater
Landon
Cox
Davis

Figure 9